Finney
Demana
Waits
Kennedy

AP* Edition

Calculus

Graphical, Numerical, Algebraic

THIRD EDITION

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Ross L. Finney
Franklin D. Demana — The Ohio State University
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Daniel Kennedy — Baylor School

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Foreword

This text, as the edition before it, was especially designed and written for teachers and students of Advanced Placement Calculus. Combining the scholarship of Ross Finney and Frank Demana, the technological expertise of Bert Waits, and the intimate knowledge of and experience with the Advanced Placement Program of Dan Kennedy, this text is truly unique among calculus texts. It may be used, in perfect order and without supplementation, from the first day of the course until the day of the AP* exam. Teachers who are new to teaching calculus, as well as those who are very experienced, will be amazed at the insightful and unique treatment of many topics.

The text is a perfect balance of exploration and theory. Students are asked to explore many topics before theoretical proof. The topic of slope fields, studied at the beginning of Chapter 6 when differential equations are first introduced, has been considerably expanded. Local linearity, stressed throughout the text, permits the early introduction of l'Hôpital's Rule. When the definite integral is introduced, students are first asked to find total change given over a specific period of time given a rate of change before they consider geometric applications. The section on logistic growth—so important in real-life situations—has been expanded. Functions are defined graphically, with tables, and with words as well as algebraically throughout the text. Problems and exercises throughout are based on real-life situations, and many are similar to questions appearing on the AP* exams. The series chapter uses technology to enhance understanding. This is a brilliant approach, and is the way that series should be presented. Students studying series from this chapter will gain a unique and thorough understanding of the topic. This textbook is one of a very few that teaches what conditional convergence means. Chapter 10, *Parametric, Vector, and Polar Functions*, covers vectors of two dimensions, and is perfect for students of Calculus BC. This chapter teaches exactly what the AP* student is expected to know about vector functions.

Ross Finney has passed away since this new edition was started, but his influence and scholarship are still keenly felt in the text. Throughout his life, Ross was always a master teacher, but even he was amazed at the insight and brilliance of the team of Dan, Frank, and Bert. This new edition is well prepared to take student and teacher on their journey through AP* Calculus, and I recommend it with the highest enthusiasm. There is no more comfortable, complete conveyance available anywhere.

—*Judith Broadwin*

Judy Broadwin taught AP Calculus at Jericho High School for many years. In addition, she was a reader, table leader, and eventually BC Exam leader of the AP* exam. She was a member to the Development Committee for AP* Calculus during the years that the AP* course descriptions were undergoing significant change. Judy now teaches calculus at Baruch College of the City of New York.*

Contents

Every section throughout the book also includes "Exploration" and "Extending the Ideas" features which follow the exercises.

About the Authors

Ross L. Finney

Ross Finney received his undergraduate degree and Ph.D. from the University of Michigan at Ann Arbor. He taught at the University of Illinois at Urbana–Champaign from 1966 to 1980 and at the Massachusetts Institute of Technology (MIT) from 1980 to 1990. Dr. Finney worked as a consultant for the Educational Development Center in Newton, Massachusetts. He directed the Undergraduate Mathematics and its Applications Project (UMAP) from 1977 to 1984 and was founding editor of the *UMAP Journal*. In 1984, he traveled with a Mathematical Association of America (MAA) delegation to China on a teacher education project through People to People International.

Dr. Finney coauthored a number of Addison-Wesley textbooks, including *Calculus; Calculus and Analytic Geometry; Elementary Differential Equations with Linear Algebra*; and *Calculus for Engineers and Scientists*. Dr. Finney's coauthors were deeply saddened by the death of their colleague and friend Ross Finney on August 4, 2000.

Franklin D. Demana

Frank Demana received his master's degree in mathematics and his Ph.D. from Michigan State University. Currently, he is Professor Emeritus of Mathematics at The Ohio State University. As an active supporter of the use of technology to teach and learn mathematics, he is cofounder of the national Teachers Teaching with Technology (T^3) professional development program. He has been the director and codirector of more than \$10 million of National Science Foundation (NSF) and foundational grant activities. He is currently a co-principal investigator on a \$3 million grant from the U.S. Department of Education Mathematics and Science Educational Research program awarded to The Ohio State University. Along with frequent presentations at professional meetings, he has published a variety of articles in the areas of computer- and calculator-enhanced mathematics instruction. Dr. Demana is also cofounder (with Bert Waits) of the annual International Conference on Technology in Collegiate Mathematics (ICTCM). He is co-recipient of the 1997 Glenn Gilbert National Leadership Award presented by the National Council of Supervisors of Mathematics, and co-recipient of the 1998 Christofferson-Fawcett Mathematics Education Award presented by the Ohio Council of Teachers of Mathematics.

Dr. Demana coauthored *Precalculus: Graphical, Numerical, Algebraic; Essential Algebra: A Calculator Approach; Transition to College Mathematics; College Algebra and Trigonometry: A Graphing Approach; College Algebra: A Graphing Approach; Precalculus: Functions and Graphs*; and *Intermediate Algebra: A Graphing Approach*.

Bert K. Waits

Bert Waits received his Ph.D. from The Ohio State University and is currently Professor Emeritus of Mathematics there. Dr. Waits is cofounder of the national Teachers Teaching with Technology (T^3) professional development program, and has been codirector or principal investigator on several large National Science Foundation projects. Dr. Waits has published articles in more than 50 nationally recognized professional journals. He frequently gives invited lectures, workshops, and minicourses at national meetings of the MAA and the National Council of Teachers of Mathematics (NCTM) on how to use computer technology to enhance the teaching and learning of mathematics. He has given invited presentations at the International Congress on Mathematical Education (ICME-6, -7, and -8) in Budapest (1988), Quebec (1992), and Seville (1996). Dr. Waits is co-recipient of the 1997 Glenn Gilbert National Leadership Award presented by the National Council of Supervisors of Mathematics, and is the cofounder (with Frank Demana) of the ICTCM. He is also co-recipient of the 1998 Christofferson-Fawcett Mathematics Education Award presented by the Ohio Council of Teachers of Mathematics.

Dr. Waits coauthored *Precalculus: Graphical, Numerical, Algebraic; College Algebra and Trigonometry: A Graphing Approach; College Algebra: A Graphing Approach; Precalculus: Functions and Graphs*; and *Intermediate Algebra: A Graphing Approach*.

Daniel Kennedy

Dan Kennedy received his undergraduate degree from the College of the Holy Cross and his master's degree and Ph.D. in mathematics from the University of North Carolina at Chapel Hill. Since 1973 he has taught mathematics at the Baylor School in Chattanooga, Tennessee, where he holds the Cartter Lupton Distinguished Professorship. Dr. Kennedy became an Advanced Placement Calculus reader in 1978, which led to an increasing level of involvement with the program as workshop consultant, table leader, and exam leader. He joined the Advanced Placement Calculus Test Development Committee in 1986, then in 1990 became the first high school teacher in 35 years to chair that committee. It was during his tenure as chair that the program moved to require graphing calculators and laid the early groundwork for the 1998 reform of the Advanced Placement Calculus curriculum. The author of the 1997 *Teacher's Guide—AP® Calculus*, Dr. Kennedy has conducted more than 50 workshops and institutes for high school calculus teachers. His articles on mathematics teaching have appeared in the *Mathematics Teacher* and the *American Mathematical Monthly*, and he is a frequent speaker on education reform at professional and civic meetings. Dr. Kennedy was named a Tandy Technology Scholar in 1992 and a Presidential Award winner in 1995.

Dr. Kennedy coauthored *Precalculus: Graphical, Numerical, Algebraic; Prentice Hall Algebra I; Prentice Hall Geometry*; and *Prentice Hall Algebra 2*.

To the Teacher

The main goal of this third edition is to realign the content with the changes in the Advanced Placement (AP*) calculus syllabus and the new type of AP* exam questions. We have also more carefully connected examples and exercises and updated the data used in examples and exercises. Cumulative Quick Quizzes are now provided two or three times in each chapter.

The course outlines for AP* Calculus reflect changes in the goals and philosophy of calculus courses now being taught in colleges and universities. The following objectives reflect the goals of the curriculum.

- Students should understand the meaning of the derivative in terms of rate of change and local linear approximations.
- Students should be able to work with functions represented graphically, numerically, analytically, or verbally, and should understand the connections among these representations.
- Students should understand the meaning of the definite integral both as a limit of Riemann sums and as a net accumulation of a rate of change, and understand the relationship between the derivative and integral.
- Students should be able to model problem situations with functions, differential equations, or integrals, and communicate both orally and in written form.
- Students should be able to represent differential equations with slope fields, solve separable differential equations analytically, and solve differential equations using numerical techniques such as Euler's method.
- Students should be able to interpret convergence and divergence of series using technology, and to use technology to help solve problems. They should be able to represent functions with series and find the Lagrange error bound for Taylor polynomials.

This revision of Finney/Thomas/Demana/Waits *Calculus* completely supports the content, goals, and philosophy of the new advanced placement calculus course description.

Calculus is explored through the interpretation of graphs and tables as well as analytic methods (multiple representation of functions). Derivatives are interpreted as rates of change and local linear approximation. Local linearity is used throughout the book. The definite integral is interpreted as total change over a specific interval and as a limit of Riemann sums. Problem situations are modeled with integrals. Chapter 6 focuses on the use of differential equations to model problems. We interpret differential equations using slope fields and then solve them analytically or numerically. Convergence and divergence of series are interpreted graphically and the Lagrange error bound is used to measure the accuracy of approximating functions with Taylor polynomials.

The use of technology is integrated throughout the book to provide a balanced approach to the teaching and learning of calculus that involves algebraic, numerical, graphical, and verbal methods (the rule of four). Students are expected to use a multirepresentational approach to investigate and solve problems, to write about their conclusions, and often to work in groups to communicate mathematics orally. This book reflects what we have learned about the appropriate use of technology in the classroom during the last decade.

The visualizations and technological explorations pioneered by Demana and Waits are incorporated throughout the book. A steady focus on the goals of the advanced placement calculus curriculum has been skillfully woven into the material by Kennedy, a master high school calculus teacher. Suggestions from numerous teachers have helped us shape this modern, balanced, technological approach to the teaching and learning of calculus.

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CHANGES FOR THIS EDITION

The course descriptions for the two Advanced Placement courses (Calculus AB and Calculus BC) have changed over the years to respond to new technology and to new points of emphasis in college and university courses. The updated editions of this textbook have consistently responded to those changes to make it easier for students and teachers to adjust. This latest edition contains significantly enhanced coverage of the following topics:

- Slope fields, now a topic for both AB and BC students, are studied in greater depth and are used to visualize differential equations from the beginning.
- Euler's method, currently a BC topic, is used as a numerical technique (with multiple examples) for solving differential equations using the insights gained from slope fields.
- Local linearity, a point of emphasis in previous editions but now more important than ever for understanding various applications of the derivative, is now a thread running throughout the book.
- More examples and exercises have been added to illustrate the connections between the graph of a function and the graph of its derivative (or the graph of f and a function defined as an integral of f).
- The logistic differential equation, a BC topic that is covered weakly in most textbooks despite its many applications, now has its own section.

Similarly, the coverage of some other topics has been trimmed to reflect the intent of their inclusion in the AP* courses:

- The use of partial fractions for finding antiderivatives has been narrowed to distinct linear factors in the denominator and has been more directly linked to the logistic differential equation;
- The treatment of vector calculus has been revised to focus on planar motion problems, which are easily solved using earlier results componentwise;
- The treatment of polar functions has been narrowed to the polar topics in the BC course description and has been linked more directly to the treatment of parametric functions.

Moreover, this latest edition continues to explore the ways teachers and students can use graphing calculator technology to enhance their understanding of calculus topics.

This edition of the text also includes new features to further assist students in their study of calculus:

- **What You'll Learn About... and Why** introduces the big ideas in each section and explains their purpose.
- At the end of each example students are encouraged to **Now Try** a related exercise at the end of the section to check their comprehension.
- A **Quick Quiz for AP* Preparation** appears every few sections, requiring students to answer questions about topics covered in multiple sections, to assist them in obtaining a conceptual understanding of the material.
- Each exercise set includes a group of **Standardized Test Questions**. Additionally, an **AP* Examination Preparation** appears at the end of each set of chapter review exercises.

For further information about new and continuing features, please consult the *To the Student* material.

CONTINUING FEATURES

Balanced Approach

A principal feature of this edition is the balance attained among the rule of four: analytic/algebraic, numerical, graphical, and verbal methods of representing problems. We believe that students must value all of these methods of representation, understand how they are connected in a given problem, and learn how to choose the one(s) most appropriate for solving a particular problem.

The Rule of Four

In support of the rule of four, we use a variety of techniques to solve problems. For instance, we obtain solutions algebraically or analytically, support our results graphically or numerically with technology, and then interpret the result in the original problem context. We have written exercises where students are asked to solve problems by one method and then support or confirm their solutions by using another method. We want students to understand that technology can be used to support (but not prove) results, and that algebraic or analytic techniques are needed to prove results. We want students to understand that mathematics provides the foundation that allows us to use technology to solve problems.

Applications

The text includes a rich array of interesting applications from biology, business, chemistry, economics, engineering, finance, physics, the social sciences, and statistics. Some applications are based on real data from cited sources. Students are exposed to functions as mechanisms for modeling data and learn about how various functions can model real-life problems. They learn to analyze and model data, represent data graphically, interpret from graphs, and fit curves. Additionally, the tabular representations of data presented in the text highlight the concept that a function is a correspondence between numerical variables, helping students to build the connection between the numbers and the graphs.

Explorations

Students are expected to be actively involved in understanding calculus concepts and solving problems. Often the explorations provide a guided investigation of a concept. The explorations help build problem-solving ability by guiding students to develop a mathematical model of a problem, solve the mathematical model, support or confirm the solution, and interpret the solution. The ability to communicate their understanding is just as important to the learning process as reading or studying, not only in mathematics but in every academic pursuit. Students can gain an entirely new perspective on their knowledge when they explain what they know in writing.

Graphing Utilities

The book assumes familiarity with a graphing utility that will produce the graph of a function within an arbitrary viewing window, find the zeros of a function, compute the derivative of a function numerically, and compute definite integrals numerically. Students are expected to recognize that a given graph is reasonable, identify all the important characteristics of a graph, interpret those characteristics, and confirm them using analytic methods. Toward that end, most graphs appearing in this book resemble students' actual grapher output or suggest hand-drawn sketches. This is one of the first calculus textbooks to take full advantage of graphing calculators, philosophically restructuring the course to teach new things in new ways to achieve new understanding, while (courageously) abandoning some old things and old ways that are no longer serving a purpose.

Exercise Sets

The exercise sets were revised extensively for this edition, including many new ones. There are nearly 4,000 exercises, with more than 80 Quick Quiz exercises and 560 Quick Review exercises. The different types of exercises included are:

Algebraic and analytic manipulation

Interpretation of graphs

Graphical representations

Numerical representations

Explorations

Writing to learn

Group activities

Data analyses

Descriptively titled applications

Extending the ideas

Each exercise set begins with the Quick Review feature, which can be used to introduce lessons, support Examples, and review prerequisite skills. The exercises that follow are graded from routine to challenging. An additional block of exercises, Extending the Ideas, may be used in a variety of ways, including group work. We also provide Review Exercises and AP* Examination Preparation at the end of each chapter.

SUPPLEMENTS AND RESOURCES

For the Student

Student Edition, ISBN 0-13-201408-4

Preparing for the Calculus AP Exam,* ISBN 0-321-33574-0

- Introduction to the AP* AB and BC Calculus Exams
- Precalculus Review of Calculus Prerequisites
- Review of AP* Calculus AB and Calculus BC Topics
- Practice Exams
- Answers and Solutions

Student Practice Workbook, ISBN 0-13-201411-4

- New examples that parallel key examples from each section in the book are provided along with a detailed solution
- Related practice problems follow each example

Texas Instruments Graphing Calculator Manual, ISBN 0-13-201415-7

- An introduction to Texas Instruments' graphing calculators, as they are used for calculus
- Features the TI-84 Plus Silver Edition, the TI-86, and the TI-89 Titanium. The keystrokes, menus and screens for the TI-83 Plus, TI-83 Plus Silver Edition, and the TI-84 Plus are similar to the TI-84 Plus Silver Edition and the TI-89, TI-92 Plus, and Voyage™ 200 are similar to the TI-89 Titanium.

For the Teacher

Annotated Teacher Edition, ISBN 0-13-201409-2

- Answers included on the same page as the problem appears, for most exercises

- Solutions to Chapter Opening Problems, Teaching Notes, Common Errors, Notes on Examples and Exploration Extensions, and Assignment Guide included at the beginning of the book.

Teacher's AP Correlations and Preparation Guide,* 0-13-201413-0

- Calculus AB/BC topic correlations, Pacing Guides for AB/BC, Assignment Guides, Concepts Worksheets, Group Activity Explorations, Sample Tests, and Answers

Assessment Resources, 0-13-201412-2

- Chapter quizzes, chapter tests, semester tests, final tests, and alternate assessments, along with all answers

Solutions Manual, ISBN 0-13-201414-9

- Complete solutions for Quick Reviews, Exercises, Explorations, and Chapter Reviews

Transparencies, ISBN 0-13-201410-6

- Full color transparencies for key figures from the text

Technology Resources

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MathXL® is a powerful online homework, tutorial, and assessment system that accompanies our textbooks in mathematics or statistics. With MathXL, instructors can create, edit, and assign online homework and tests using algorithmically generated exercises correlated at the objective level to the textbook. They can also create and assign their own online exercises and import TestGen tests for added flexibility. All student work is tracked in MathXL's online gradebook. Students can take chapter tests in MathXL and receive personalized study plans based on their test results. The study plan diagnoses weaknesses and links students directly to tutorial exercises for the objectives they need to study and retest. Students can also access supplemental animations and video clips directly from selected exercises. For more information, visit our Web site at www.mathxl.com, or contact your local sales representative.

InterAct Math Tutorial Web site, www.interactmath.com

Get practice and tutorial help online! This interactive tutorial Web site provides algorithmically generated practice exercises that correlate directly to the exercises in the textbook. Students can retry an exercise as many times as they like with new values each time for unlimited practice and mastery. Every exercise is accompanied by an interactive guided solution that provides helpful feedback for incorrect answers, and students can also view a worked-out sample problem that steps them through an exercise similar to the one they're working on.

Video Lectures on CD, ISBN 0-13-2030709-5

The video lectures feature engaging mathematics instructors who present comprehensive coverage of the core topics of the text. The presentations include examples and exercises from the text and support an approach that emphasizes visualization and problem-solving.

TestGen®, ISBN 0-13-201419-X

TestGen® enables instructors to build, edit, print, and administer tests using a computerized bank of questions developed to cover all the objectives of the text. TestGen is algorithmically based, allowing instructors to create multiple but equivalent versions of the same question or test with the click of a button. Instructors can also modify test questions or add new questions by using the built-in question editor, which allows users to create graphs, import graphics, and insert math notation, variable numbers, or text. Tests can be printed or administered online via the Internet or another network. TestGen comes packaged with QuizMaster, which allows students to take tests on a local area network. The software is available on a dual-platform Windows/Macintosh CD-ROM.

Presentation Express CD-ROM, ISBN 0-13-201420-3

This time saving component includes all the transparencies in PowerPoint format as well as section-by-section lecture notes for the entire book, making it easier for you to teach and to customize based on your teaching preferences. All slides can be customized and edited.

Teacher Express CD-ROM (with LessonView), ISBN 0-13-201422-X

Plan – Teach – Assess. TeacherEXPRESS is a new suite of instructional tools on CD-ROM to help teachers plan, teach, and assess at the click of a mouse. Powerful lesson planning, resource management, testing, and an interactive teacher's edition all in one place make class preparation quick and easy! Contents: Planning Express, Teacher's Edition, Program Teaching Resources, Correlations, and Links to Other Resources.

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To the AP* Student

We know that as you study for your AP* course, you're preparing along the way for the AP* exam. By tying the material in this book directly to AP* course goals and exam topics, we help you to focus your time most efficiently. And that's a good thing!

The AP* exam is an important milestone in your education. A high score will position you optimally for college acceptance—and possibly will give you college credits that put you a step ahead. Our primary commitment is to provide you with the tools you need to excel on the exam ... the rest is up to you!

Test-Taking Strategies for an Advanced Placement* Calculus Examination

You should approach the AP* Calculus Examination the same way you would any major test in your academic career. Just remember that it is a one-shot deal—you should be at your peak performance level on the day of the test. For that reason you should do everything that your "coach" tells you to do. In most cases your coach is your classroom teacher. It is very likely that your teacher has some experience, based on workshop information or previous students' performance, to share with you.

You should also analyze your own test-taking abilities. At this stage in your education, you probably know your strengths and weaknesses in test-taking situations. You may be very good at multiple choice questions but weaker in essays, or perhaps it is the other way around. Whatever your particular abilities are, evaluate them and respond accordingly. Spend more time on your weaker points. In other words, rather than spending time in your comfort zone where you need less work, try to improve your soft spots. In all cases, concentrate on clear communication of your strategies, techniques, and conclusions.

The following table presents some ideas in a quick and easy form.

General Strategies for AP* Examination Preparation

Time	Dos
Through the Year	• Register with your teacher/coordinator • Pay your fee (if applicable) on time • Take good notes • Work with others in study groups • Review on a regular basis • Evaluate your test-taking strengths and weaknesses—keep track of how successful you are when guessing
The Week Before	• Combine independent and group review • Get tips from your teacher • Do lots of mixed review problems • Check your exam date, time, and location • Review the appropriate AP* Calculus syllabus (AB or BC)
The Night Before	• Put new batteries in your calculator • Make sure your calculator is on the approved list • Lay out your clothes and supplies so that you are ready to go out the door • Do a short review • Go to bed at a reasonable hour
Exam Day	• Get up a little earlier than usual • Eat a good breakfast/lunch • Put some hard candy in your pocket in case you need an energy boost during the test • Get to your exam location 15 minutes early
Exam Night	• Relax–you earned it

Topics from the Advanced Placement* Curriculum for Calculus AB, Calculus BC

As an AP* Student, you are probably well aware of the good study habits that are needed to be a successful student in high school and college:

- attend all the classes
- ask questions (either during class or after)
- take clear and understandable notes
- make sure you understand the concepts rather than memorizing formulas
- do your homework; extend your test-prep time over several days or weeks, instead of cramming
- use all the resources—text and people—that are available to you.

No doubt this list of "good study habits" is one that you have seen or heard before. You should know that there is powerful research that suggests a few habits or routines will enable you to go beyond "knowing about" calculus, to more deeply "understanding" calculus. Here are three concrete actions for you to consider:

- Review your notes at least once a week and rewrite them in summary form.
- Verbally explain concepts (theorems, etc.) to a classmate.

- Form a study group that meets regularly to do homework and discuss reading and lecture notes.

Most of these tips boil down to one mantra, which all mathematicians believe in:

Math is not a spectator sport.

The AP* Calculus Examination is based on the following Topic Outline. For your convenience, we have noted all Calculus AB and Calculus BC objectives with clear indications of topics required only by the Calculus BC Exam. The outline cross-references each AP* Calculus objective with the appropriate section(s) of this textbook: *Calculus: Graphical, Numerical, Algebraic,* Third Edition, by Finney, Demana, Waits, and Kennedy.

Use this outline to track your progress through the AP* exam topics. Be sure to cover every topic associated with the exam you are taking. Check it off when you have studied and/or reviewed the topic.

Even as you prepare for your exam, I hope this book helps you map—and enjoy—your calculus journey!

—John Brunsting
Hinsdale Central High School

Topic Outline for AP* Calculus AB and AP* Calculus BC

(excerpted from the College Board's Course Description - Calculus: Calculus AB, Calculus BC, May 2007)

I.	Calculus Exam		Functions, Graphs, and Limits	*Calculus*
A	AB	BC	Analysis of graphs	1.2–1.6
B	AB	BC	Limits of functions (including one-sided limits)	
B1	AB	BC	An intuitive understanding of the limiting process	2.1, 2.2
B2	AB	BC	Calculating limits using algebra	2.1, 2.2
B3	AB	BC	Estimating limits from graphs or tables of data	2.1, 2.2
C	AB	BC	Asymptotic and unbounded behavior	
C1	AB	BC	Understanding asymptotes in terms of graphical behavior	2.2
C2	AB	BC	Describing asymptotic behavior in terms of limits involving infinity	2.2
C3	AB	BC	Comparing relative magnitudes of functions and their rates of change	2.2, 2.4, 8.3
D	AB	BC	Continuity as a property of functions	
D1	AB	BC	An intuitive understanding of continuity	2.3
D2	AB	BC	Understanding continuity in terms of limits	2.3
D3	AB	BC	Geometric understanding of graphs of continuous functions	2.3, 4.1–4.3
E		BC	Parametric, polar, and vector functions	10.1 10.3

II.	Calculus Exam		Derivatives	*Calculus*
A	AB	BC	Concept of the derivative	
A1	AB	BC	Derivative presented graphically, numerically, and analytically	2.4–4.5
A2	AB	BC	Derivative interpreted as an instantaneous rate of change	2.4
A3	AB	BC	Derivative defined as the limit of the difference quotient	2.4–3.1
A4	AB	BC	Relationship between differentiability and continuity	3.2
B	AB	BC	Derivative at a point	
B1	AB	BC	Slope of a curve at a point	2.4
B2	AB	BC	Tangent line to a curve at a point and local linear approximation	2.4, 4.5
B3	AB	BC	Instantaneous rate of change as the limit of average rate of change	2.4, 3.4
B4	AB	BC	Approximate rate of change from graphs and tables of values	2.4, 3.4
C	AB	BC	Derivative as a function	
C1	AB	BC	Corresponding characteristics of graphs of f and f'	3.1, 4.3

C2	AB	BC	Relationship between the increasing and decreasing behavior of f and the sign of f'	4.1, 4.3
C3	AB	BC	The Mean Value Theorem and its geometric consequences.	4.2
C4	AB	BC	Equations involving derivatives. Verbal descriptions are translated into equations involving derivatives and vice versa	3.4, 3.5, 4.6, 6.4, 6.5
D	AB	BC	Second Derivatives	
D1	AB	BC	Corresponding characteristics of graphs of f, f' and f''	4.3
D2	AB	BC	Relationship between the concavity of f and the sign of f''	4.3
D3	AB	BC	Points of inflection as places where concavity changes	4.3
E	AB	BC	Applications of derivatives	
E1	AB	BC	Analysis of curves, including the notions of monotonicity and concavity	4.1–4.3
E2		BC	Analysis of planar curves given in parametric form, polar form, and vector form, including velocity and acceleration vectors	10.1–10.3
E3	AB	BC	Optimization, both absolute (global) and relative (local) extrema	4.3, 4.4
E4	AB	BC	Modeling rates of change, including related rates problems	4.6
E5	AB	BC	Use of implicit differentiation to find the derivative of an inverse function	3.7
E6	AB	BC	Interpretation of the derivative as a rate of change in varied applied contexts, including velocity, speed, and acceleration	3.4
E7	AB	BC	Geometric interpretation of differential equations via slope fields and the relationship between slope fields and solution curves for differential equations	6.1
E8		BC	Numerical solution of differential equations using Euler's method	6.1
E9		BC	L'Hopital's Rule, including its use in determining limits and convergence of improper integrals and series	8.1, 9.5
F	AB	BC	Computation of derivatives	
F1	AB	BC	Knowledge of derivatives of basic functions, including power, exponential, logarithmic, trigonometric, and inverse trigonometric functions	3.3, 3.5, 3.8, 3.9
F2	AB	BC	Basic rules for the derivative of sums, products, and quotients of functions	3.3
F3	AB	BC	Chain rule and implicit differentiation	3.6, 3.7
F4		BC	Derivatives of parametric, polar, and vector functions	10.1–10.3

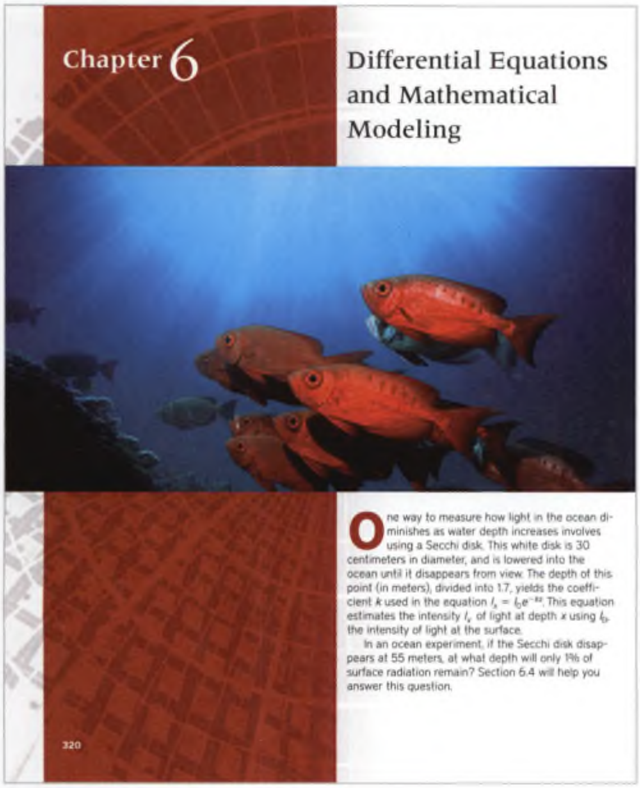
III.	**Calculus Exam**		**Integrals**	***Calculus***
A			Interpretations and properties of definite integrals	
A1	AB	BC	Definite integral as a limit of Riemann sums	5.1, 5.2
A2	AB	BC	Definite integral of the rate of change of a quantity over an interval interpreted as the change of the quantity over the closed interval $[a,b]$ of $\int f'(x)dx = f(b) - f(a)$	5.1, 5.4
A3	AB	BC	Basic properties of definite integrals (Examples include additivity and linearity.)	5.2 - 5.3
B			Applications of integrals	
B1a	AB	BC	Appropriate integrals are used in a variety of applications to model physical, biological, or economic situations. ... students should be able to adapt their knowledge and techniques. Emphasis is on using the method of setting up an approximating Riemann sum and representing its limit as a definite integral. ... specific applications should include using the integral of a rate of change to give accumulated change, finding the area of a region, the volume of a solid with known cross sections, the average value of a function, and the distance traveled by a particle along a line	5.4, 5.5, 6.4, 6.5, 7.1–7.5
B1b		BC	Appropriate integrals are used ... specific applications should include ... finding the area of a region bounded by polar curves ... and the length of a curve (including a curve given in parametric form)	7.4, 10.1, 10.3
C			Fundamental Theorem of Calculus	
C1	AB	BC	Use of the Fundamental Theorem to evaluate definite integrals	5.4
C2	AB	BC	Use of the Fundamental Theorem to represent a particular antiderivative, and the analytical and graphical analysis of functions so derived	5.4, 6.1

D			Techniques of antidifferentiation	
D1	AB	BC	Antiderivatives following directly from derivatives of basic functions	4.2, 6.1, 6.2
D2a	AB	BC	Antiderivatives by substitution of variables (including change of limits for definite integrals)	6.2
D2b		BC	Antiderivatives by ... parts, and simple partial fractions (nonrepeating linear factors only)	6.3, 6.5
D3		BC	Improper integrals (as limits of definite integrals)	8.3
E			Applications of antidifferrentiation	
E1	AB	BC	Finding specific antiderivatives using initial conditions, including applications to motion along a line	6.1, 7.1
E2	AB	BC	Solving separable differential equations and using them in modeling In particular, studying the equations $y' = ky$ and exponential growth	6.4
E3		BC	Solving logistic differential equations and using them in modeling	6.5
F			Numerical approximations to definite integrals	
F1	AB	BC	Use of Riemann and trapezoidal sums to approximate definite integrals of functions represented algebraically, graphically, and by tables of values	5.2, 5.5

IV.	**Calculus Exam**		**Polynomial Approximations and Series**	***Calculus***
A			Concept of series	
A1		BC	A series is defined as a sequence of partial sums, and convergence is defined in terms of the limit of the sequence of partial sums. Technology can be used to explore convergence or divergence	9.1
B			Series of constants	
B1		BC	Motivating examples, including decimal expansion	9.1
B2		BC	Geometric series with applications	9.1
B3		BC	The harmonic series	9.5
B4		BC	Alternating series with error bound	9.5
B5		BC	Terms of series as areas of rectangles and their relationship to improper integrals, including the integral test and its use in testing the convergence of p-series	9.5
B6		BC	The ratio test for convergence or divergence	9.4
B7		BC	Comparing series to test for convergence and divergence	9.4
C			Taylor series	
C1		BC	Taylor polynomial approximation with graphical demonstration of convergence (For example, viewing graphs of various Taylor polynomials of the sine function approximating the sine curve.)	9.2
C2		BC	Maclaurin series and the general Taylor series centered at $x = a$	9.2
C3		BC	Maclaurin series for the functions e^x, $\sin x$, $\cos x$, and $1/(1 - x)$	9.2
C4		BC	Formal manipulation of Taylor series and shortcuts to computing Taylor series, including substitution, differentiation, antidifferentiation, and the formation of new series from known series	9.1, 9.2
C5		BC	Functions defined by power series	9.1, 9.2
C6		BC	Radius and interval of convergence of power series	9.1, 9.4, 9.5
C7		BC	Lagrange error bound for Taylor polynomials	9.3

Using the Book for Maximum Effectiveness

So, how can this book help you to join in the game of mathematics for a winning future? Let us show you some unique tools that we have included in the text to help prepare you not only for the AP* Calculus exam, but also for success beyond this course.



Chapter 6

Differential Equations and Mathematical Modeling

One way to measure how light in the ocean diminishes as water depth increases involves using a Secchi disk. This white disk is 30 centimeters in diameter, and is lowered into the ocean until it disappears from view. The depth of this point (in meters), divided into 1.7, yields the coefficient k used in the equation $I_x = I_0 e^{-kx}$. This equation estimates the intensity I_x of light at depth x using I_0, the intensity of light at the surface.

In an ocean experiment, if the Secchi disk disappears at 55 meters, at what depth will only 1% of surface radiation remain? Section 6.4 will help you answer this question.

320

Chapter Openers provide a motivating photograph and application to show you an example that illustrates the relevance of what you'll be learning in the chapter.

A **Chapter Overview** then follows to give you a sense of what you are going to learn. This overview provides a roadmap of the chapter as well as tells how the different topics in the chapter are connected under one big idea. It is always helpful to remember that mathematics isn't modular, but interconnected, and that the different skills you are learning throughout the course build on one another to help you understand more complex concepts.

Chapter 6 Overview

One of the early accomplishments of calculus was predicting the future position of a planet from its present position and velocity. Today this is just one of a number of occasions on which we deduce everything we need to know about a function from one of its known values and its rate of change. From this kind of information, we can tell how long a sample of radioactive polonium will last; whether, given current trends, a population will grow or become extinct; and how large major league baseball salaries are likely to be in the year 2010. In this chapter, we examine the analytic, graphical, and numerical techniques on which such predictions are based.

6.1

What you'll learn about

- Differential Equations
- Slope Fields
- Euler's Method

. . . and why

Differential equations have always been a prime motivation for the study of calculus and remain so to this day.

Similarly, the **What you'll learn about...and why** feature gives you the big ideas in each section and explains their purpose. You should read this as you begin the section and always review it after you have completed the section to make sure you understand all of the key topics that you have just studied.

Differential Equation Mode

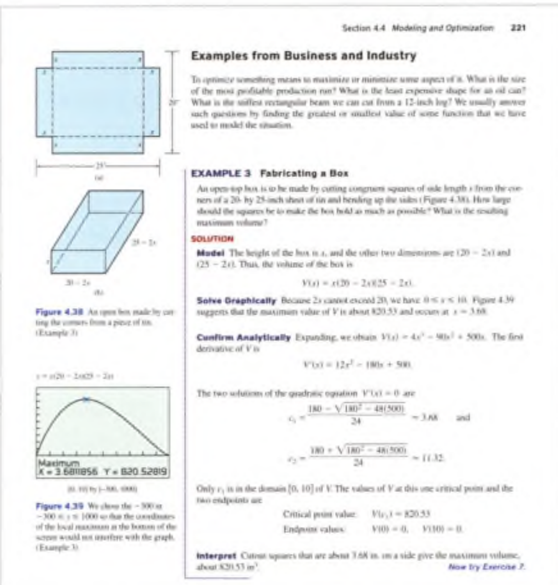
If your calculator has a *differential equation mode* for graphing, it is intended for graphing slope fields. The usual "Y=" turns into a "$dy/dx =$" screen, and you can enter a function of x and/or y. The grapher draws a slope field for the differential equation when you press the GRAPH button.

Margin Notes appear throughout the book on various topics. Some notes provide more information on a key concept or an example. Other notes offer practical advice on using your graphing calculator to obtain the most accurate results.

Brief **Historical Notes** present the stories of people and the research that they have done to advance the study of mathematics. Reading these notes will often provide you with additional insight for solving problems that you can use later when doing the homework or completing the AP* Exam.

Charles Richard Drew
(1904–1950)

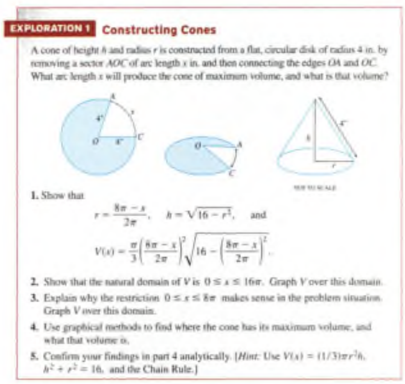
Millions of people are alive today because of Charles Drew's pioneering work on blood plasma and the preservation of human blood for transfusion. After directing the Red Cross program that collected plasma for the Armed Forces in World War II, Dr. Drew went on to become Head of Surgery at Howard University and Chief of Staff at Freedmen's Hospital in Washington, D.C.



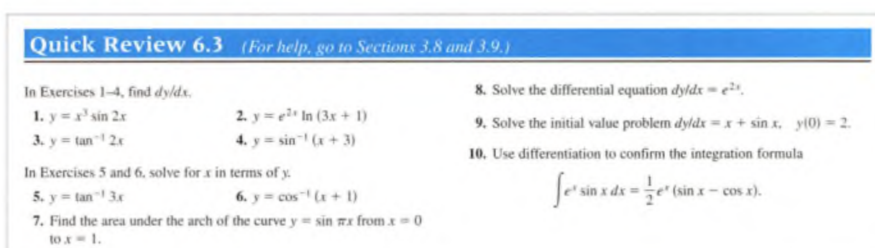
Many examples include solutions to **Solve Algebraically, Solve Graphically,** or **Solve Numerically**. You should be able to use different approaches for finding solutions to problems. For instance, you would obtain a solution algebraically when that is the most appropriate technique to use, and you would obtain solutions graphically or numerically when algebra is difficult or impossible to use. We urge you to solve problems by one method, then support or confirm your solution by using another method, and finally, interpret the results in the context of the problem. Doing so reinforces the idea that to understand a problem fully, you need to understand it algebraically, graphically, and numerically whenever possible.

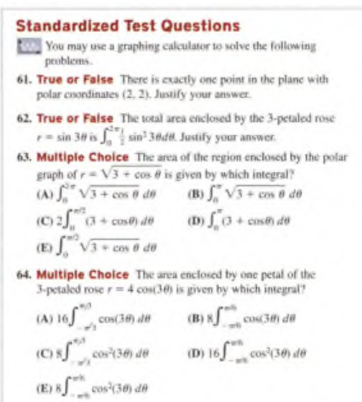
Each example ends with a suggestion to **Now Try** a related exercise. Working the suggested exercise is an easy way for you to check your comprehension of the material while reading each section, instead of waiting until the end of each section or chapter to see if you "got it." True comprehension of the textbook is essential for your success on the AP* Exam.

Explorations appear throughout the text and provide you with the perfect opportunity to become an active learner and discover mathematics on your own. Honing your critical thinking and problem-solving skills will ultimately benefit you on all of your AP* Exams.

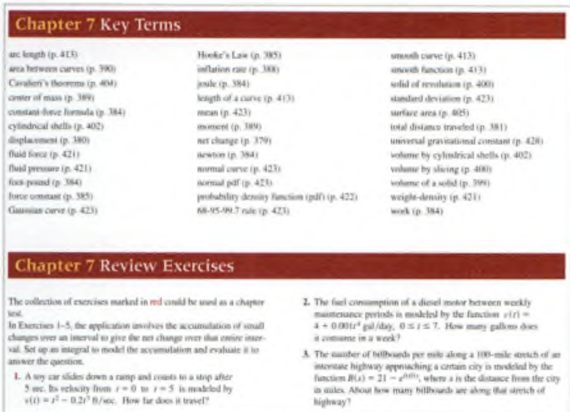


Each exercise set begins with a **Quick Review** to help you review skills needed in the exercise set, reminding you again that mathematics is not modular. Each Quick Review includes section references to show where these skills were covered earlier in the text. If you find these problems overly challenging, you should go back through the book and your notes to review the material covered in previous chapters. Remember, you need to *understand* the material from the *entire* calculus course for the AP* Calculus Exam, not just memorize the concepts from the last part of the course.



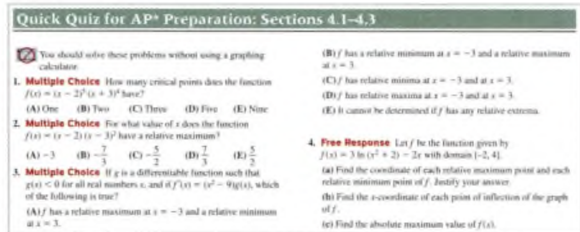


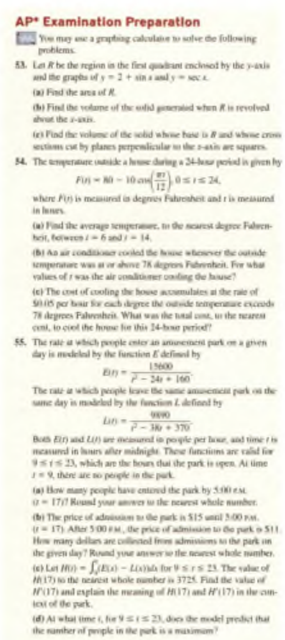
Along with the standard types of exercises, including skill-based, application, writing, exploration, and extension questions, each exercise set includes a group of **Standardized Test Questions**. Each group includes two true-false with justifications and four multiple-choice questions, with instructions about the permitted use of your graphing calculator.



Each chapter concludes with a list of **Key Terms**, with references back to where they are covered in the chapter, as well as **Chapter Review Exercises** to check your comprehension of the chapter material.

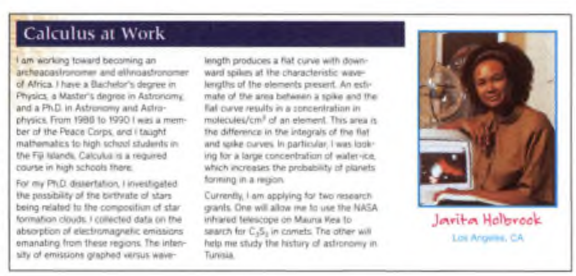
The **Quick Quiz for AP* Preparation** provides another opportunity to review your understanding as you progress through each chapter. A quiz appears after every two or three sections and asks you to answer questions about topics covered in those sections. Each quiz contains three multiple-choice questions and one free-response question of the AP* type. This continual reinforcement of ideas steers you away from rote memorization and toward the conceptual understanding needed for the AP* Calculus Exam.





An **AP* Examination Preparation** section appears at the end of each set of chapter review exercises and includes three free-response questions of the AP* type. This set of questions, which also may or may not permit the use of your graphing calculator, gives you additional opportunity to practice skills and problem-solving techniques needed for the AP* Calculus Exam.

Calculus at Work features individuals who are using calculus in their jobs, providing you with some insight as to when you will use calculus in your careers. Some of the applications of calculus they encounter are mentioned throughout the text.



In addition to this text, *Preparing for the AP* Calculus AB or BC Examinations*, written by experienced AP* teachers, is also available to help you prepare for the AP* Calculus Exam. What does it include?

- **Text-specific correlations** between key AP* test topics and *Calculus: Graphical, Numerical, Algebraic*
- Reinforcement of the important **connections** between what you'll learn and what you'll be tested on in May
- 2 full **sample AB exams** & 2 **sample BC exams** including answers and explanation
- Test Taking **strategies**

You can order *Preparing for the AP* Calculus AB or BC Examinations* by going online to PHSchool.com/catalog or calling 1-800-848-9500 and requesting ISBN 0-321-33574-0.

Acknowledgments

Many individuals contributed to the development of this textbook that is written especially for Advanced Placement Calculus teachers. To those of you who have labored on this and previous editions of this text, we offer our deepest gratitude. We also extend our sincere thanks to the dedicated users and reviewers of the previous editions of this textbook whose invaluable insight forms the heart of each textbook revision. We apologize for any omissions:

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Chapter 1

Prerequisites for Calculus

Exponential functions are used to model situations in which growth or decay change dramatically. Such situations are found in nuclear power plants, which contain rods of plutonium-239; an extremely toxic radioactive isotope.

Operating at full capacity for one year, a 1,000 megawatt power plant discharges about 435 lb of plutonium-239. With a half-life of 24,400 years, how much of the isotope will remain after 1,000 years? This question can be answered with the mathematics covered in Section 1.3.

Chapter 1 Overview

This chapter reviews the most important things you need to know to start learning calculus. It also introduces the use of a graphing utility as a tool to investigate mathematical ideas, to support analytic work, and to solve problems with numerical and graphical methods. The emphasis is on functions and graphs, the main building blocks of calculus.

Functions and parametric equations are the major tools for describing the real world in mathematical terms, from temperature variations to planetary motions, from brain waves to business cycles, and from heartbeat patterns to population growth. Many functions have particular importance because of the behavior they describe. Trigonometric functions describe cyclic, repetitive activity; exponential, logarithmic, and logistic functions describe growth and decay; and polynomial functions can approximate these and most other functions.

1.1 Lines

What you'll learn about

- Increments
- Slope of a Line
- Parallel and Perpendicular Lines
- Equations of Lines
- Applications

. . . and why

Linear equations are used extensively in business and economic applications.

Increments

One reason calculus has proved to be so useful is that it is the right mathematics for relating the rate of change of a quantity to the graph of the quantity. Explaining that relationship is one goal of this book. It all begins with the slopes of lines.

When a particle in the plane moves from one point to another, the net changes or *increments* in its coordinates are found by subtracting the coordinates of its starting point from the coordinates of its stopping point.

DEFINITION Increments

If a particle moves from the point (x_1, y_1) to the point (x_2, y_2), the **increments** in its coordinates are

$$\Delta x = x_2 - x_1 \quad \text{and} \quad \Delta y = y_2 - y_1.$$

The symbols Δx and Δy are read "delta x" and "delta y." The letter Δ is a Greek capital d for "difference." Neither Δx nor Δy denotes multiplication; Δx is not "delta times x" nor is Δy "delta times y."

Increments can be positive, negative, or zero, as shown in Example 1.

EXAMPLE 1 Finding Increments

The coordinate increments from $(4, -3)$ to $(2, 5)$ are

$$\Delta x = 2 - 4 = -2, \qquad \Delta y = 5 - (-3) = 8.$$

From $(5, 6)$ to $(5, 1)$, the increments are

$$\Delta x = 5 - 5 = 0, \qquad \Delta y = 1 - 6 = -5.$$

Now try Exercise 1.

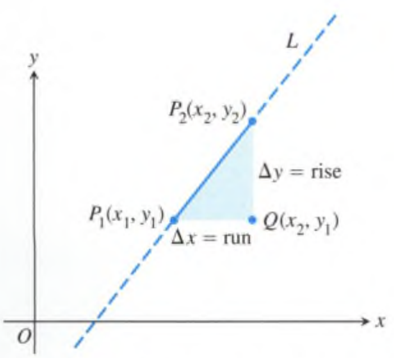


Figure 1.1 The slope of line L is $m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x}$.

Slope of a Line

Each nonvertical line has a *slope,* which we can calculate from increments in coordinates.

Let L be a nonvertical line in the plane and $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ two points on L (Figure 1.1). We call $\Delta y = y_2 - y_1$ the **rise** from P_1 to P_2 and $\Delta x = x_2 - x_1$ the **run** from

P_1 to P_2. Since L is not vertical, $\Delta x \neq 0$ and we define the slope of L to be the amount of rise per unit of run. It is conventional to denote the slope by the letter m.

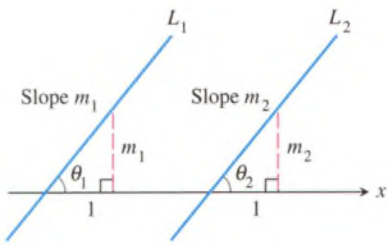


Figure 1.2 If $L_1 \parallel L_2$, then $\theta_1 = \theta_2$ and $m_1 = m_2$. Conversely, if $m_1 = m_2$, then $\theta_1 = \theta_2$ and $L_1 \parallel L_2$.

DEFINITION Slope

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be points on a nonvertical line, L. The **slope** of L is

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

A line that goes uphill as x increases has a positive slope. A line that goes downhill as x increases has a negative slope. A horizontal line has slope zero since all of its points have the same y-coordinate, making $\Delta y = 0$. For vertical lines, $\Delta x = 0$ and the ratio $\Delta y/\Delta x$ is undefined. We express this by saying that vertical lines *have no slope.*

Parallel and Perpendicular Lines

Parallel lines form equal angles with the x-axis (Figure 1.2). Hence, nonvertical parallel lines have the same slope. Conversely, lines with equal slopes form equal angles with the x-axis and are therefore parallel.

If two nonvertical lines L_1 and L_2 are perpendicular, their slopes m_1 and m_2 satisfy $m_1 m_2 = -1$, so each slope is the *negative reciprocal* of the other:

$$m_1 = -\frac{1}{m_2}, \qquad m_2 = -\frac{1}{m_1}.$$

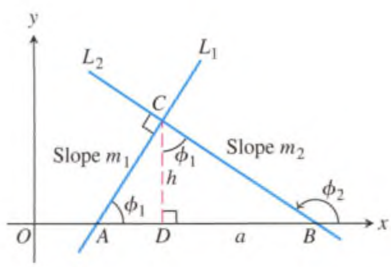


Figure 1.3 $\triangle ADC$ is similar to $\triangle CDB$. Hence ϕ_1 is also the upper angle in $\triangle CDB$, where $\tan \phi_1 = a/h$.

The argument goes like this: In the notation of Figure 1.3, $m_1 = \tan\phi_1 = a/h$, while $m_2 = \tan\phi_2 = -h/a$. Hence, $m_1 m_2 = (a/h)(-h/a) = -1$.

Equations of Lines

The vertical line through the point (a, b) has equation $x = a$ since every x-coordinate on the line has the value a. Similarly, the horizontal line through (a, b) has equation $y = b$.

EXAMPLE 2 Finding Equations of Vertical and Horizontal Lines

The vertical and horizontal lines through the point (2, 3) have equations $x = 2$ and $y = 3$, respectively (Figure 1.4). *Now try Exercise 9.*

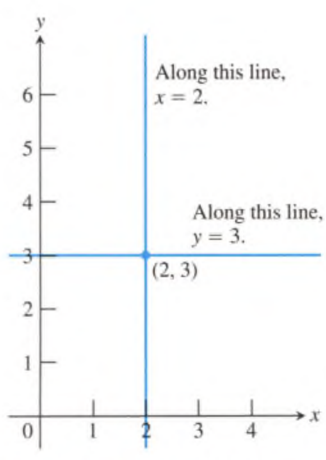


Figure 1.4 The standard equations for the vertical and horizontal lines through the point (2, 3) are $x = 2$ and $y = 3$. (Example 2)

We can write an equation for any nonvertical line L if we know its slope m and the coordinates of one point $P_1(x_1, y_1)$ on it. If $P(x, y)$ is *any* other point on L, then

$$\frac{y - y_1}{x - x_1} = m,$$

so that

$$y - y_1 = m(x - x_1) \quad \text{or} \quad y = m(x - x_1) + y_1.$$

DEFINITION Point-Slope Equation

The equation

$$y = m(x - x_1) + y_1$$

is the **point-slope equation** of the line through the point (x_1, y_1) with slope m.

EXAMPLE 3 Using the Point-Slope Equation

Write the point-slope equation for the line through the point (2, 3) with slope $-3/2$.

SOLUTION

We substitute $x_1 = 2$, $y_1 = 3$, and $m = -3/2$ into the point-slope equation and obtain

$$y = -\frac{3}{2}(x - 2) + 3 \quad \text{or} \quad y = -\frac{3}{2}x + 6.$$

Now try Exercise 13.

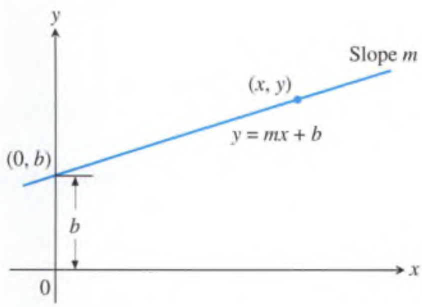


Figure 1.5 A line with slope m and y-intercept b.

The y-coordinate of the point where a nonvertical line intersects the y-axis is the **y-intercept** of the line. Similarly, the x-coordinate of the point where a nonhorizontal line intersects the x-axis is the **x-intercept** of the line. A line with slope m and y-intercept b passes through $(0, b)$ (Figure 1.5), so

$$y = m(x - 0) + b, \quad \text{or, more simply,} \quad y = mx + b.$$

DEFINITION Slope-Intercept Equation

The equation

$$y = mx + b$$

is the **slope-intercept equation** of the line with slope m and y-intercept b.

EXAMPLE 4 Writing the Slope-Intercept Equation

Write the slope-intercept equation for the line through $(-2, -1)$ and $(3, 4)$.

SOLUTION

The line's slope is

$$m = \frac{4 - (-1)}{3 - (-2)} = \frac{5}{5} = 1.$$

We can use this slope with either of the two given points in the point-slope equation. For $(x_1, y_1) = (-2, -1)$, we obtain

$$y = 1 \cdot (x - (-2)) + (-1)$$
$$y = x + 2 + (-1)$$
$$y = x + 1.$$

Now try Exercise 17.

If A and B are not both zero, the graph of the equation $Ax + By = C$ is a line. Every line has an equation in this form, even lines with undefined slopes.

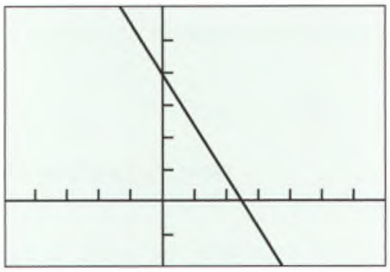
DEFINITION General Linear Equation

The equation

$$Ax + By = C \quad (A \text{ and } B \text{ not both } 0)$$

is a **general linear equation** in x and y.

$y = -\frac{8}{5}x + 4$



[–5, 7] by [–2, 6]

Figure 1.6 The line $8x + 5y = 20$. (Example 5)

Although the general linear form helps in the quick identification of lines, the slope-intercept form is the one to enter into a calculator for graphing.

EXAMPLE 5 Analyzing and Graphing a General Linear Equation

Find the slope and y-intercept of the line $8x + 5y = 20$. Graph the line.

SOLUTION

Solve the equation for y to put the equation in slope-intercept form:

$$8x + 5y = 20$$
$$5y = -8x + 20$$
$$y = -\frac{8}{5}x + 4$$

This form reveals the slope ($m = -8/5$) and y-intercept ($b = 4$), and puts the equation in a form suitable for graphing (Figure 1.6).

Now try Exercise 27.

EXAMPLE 6 Writing Equations for Lines

Write an equation for the line through the point $(-1, 2)$ that is **(a)** parallel, and **(b)** perpendicular to the line L: $y = 3x - 4$.

SOLUTION

The line L, $y = 3x - 4$, has slope 3.

(a) The line $y = 3(x + 1) + 2$, or $y = 3x + 5$, passes through the point $(-1, 2)$, and is parallel to L because it has slope 3.

(b) The line $y = (-1/3)(x + 1) + 2$, or $y = (-1/3)x + 5/3$, passes through the point $(-1, 2)$, and is perpendicular to L because it has slope $-1/3$.

Now try Exercise 31.

EXAMPLE 7 Determining a Function

The following table gives values for the linear function $f(x) = mx + b$. Determine m and b.

x	$f(x)$
-1	$14/3$
1	$-4/3$
2	$-13/3$

SOLUTION

The graph of f is a line. From the table we know that the following points are on the line: $(-1, 14/3)$, $(1, -4/3)$, $(2, -13/3)$.

Using the first two points, the slope m is

$$m = \frac{-4/3 - (14/3)}{1 - (-1)} = \frac{-6}{2} = -3.$$

So $f(x) = -3x + b$. Because $f(-1) = 14/3$, we have

$$f(-1) = -3(-1) + b$$
$$14/3 = 3 + b$$
$$b = 5/3.$$

continued

Thus, $m = -3$, $b = 5/3$, and $f(x) = -3x + 5/3$.

We can use either of the other two points determined by the table to check our work.

Now try Exercise 35.

Applications

Many important variables are related by linear equations. For example, the relationship between Fahrenheit temperature and Celsius temperature is linear, a fact we use to advantage in the next example.

EXAMPLE 8 Temperature Conversion

Find the relationship between Fahrenheit and Celsius temperature. Then find the Celsius equivalent of 90°F and the Fahrenheit equivalent of −5°C.

SOLUTION

Because the relationship between the two temperature scales is linear, it has the form $F = mC + b$. The freezing point of water is $F = 32°$ or $C = 0°$, while the boiling point is $F = 212°$ or $C = 100°$. Thus,

$$32 = m \cdot 0 + b \quad \text{and} \quad 212 = m \cdot 100 + b,$$

so $b = 32$ and $m = (212 - 32)/100 = 9/5$. Therefore,

$$F = \frac{9}{5}C + 32, \quad \text{or} \quad C = \frac{5}{9}(F - 32).$$

These relationships let us find equivalent temperatures. The Celsius equivalent of 90°F is

$$C = \frac{5}{9}(90 - 32) \approx 32.2°.$$

The Fahrenheit equivalent of −5°C is

$$F = \frac{9}{5}(-5) + 32 = 23°.$$

Now try Exercise 43.

Some graphing utilities have a feature that enables them to approximate the relationship between variables with a linear equation. We use this feature in Example 9.

It can be difficult to see patterns or trends in lists of paired numbers. For this reason, we sometimes begin by plotting the pairs (such a plot is called a **scatter plot**) to see whether the corresponding points lie close to a curve of some kind. If they do, and if we can find an equation $y = f(x)$ for the curve, then we have a formula that

1. summarizes the data with a simple expression, and
2. lets us predict values of y for other values of x.

The process of finding a curve to fit data is called **regression analysis** and the curve is called a **regression curve.**

There are many useful types of regression curves—power, exponential, logarithmic, sinusoidal, and so on. In the next example, we use the calculator's linear regression feature to fit the data in Table 1.1 with a line.

Table 1.1 World Population

Year	Population (millions)
1980	4454
1985	4853
1990	5285
1995	5696
2003	6305
2004	6378
2005	6450

Source: U.S. Bureau of the Census, *Statistical Abstract of the United States, 2004–2005.*

EXAMPLE 9 Regression Analysis—Predicting World Population

Starting with the data in Table 1.1, build a linear model for the growth of the world population. Use the model to predict the world population in the year 2010, and compare this prediction with the Statistical Abstract prediction of 6812 million.

continued

Why Not Round the Decimals in Equation 1 Even More?

If we do, our final calculation will be way off. Using $y = 80x - 153,849$, for instance, gives $y = 6951$ when $x = 2010$, as compared to $y = 6865$, an increase of 86 million. The rule is: *Retain all decimal places while working a problem. Round only at the end.* We rounded the coefficients in Equation 1 enough to make it readable, but not enough to hurt the outcome. However, we knew how much we could safely round *only from first having done the entire calculation with numbers unrounded.*

Rounding Rule

Round your answer as appropriate, but do not round the numbers in the calculations that lead to it.

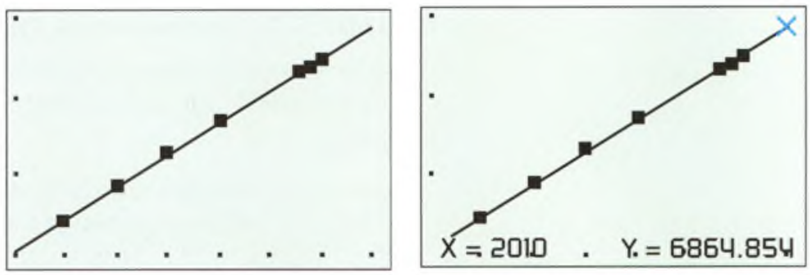
SOLUTION

Model Upon entering the data into the grapher, we find the regression equation to be approximately

$$y = 79.957x - 153848.716, \tag{1}$$

where x represents the year and y the population *in millions.*

Figure 1.7a shows the scatter plot for Table 1.1 together with a graph of the regression line just found. You can see how well the line fits the data.



[1975, 2010] by [4000, 7000]
(a)

[1975, 2010] by [4000, 7000]
(b)

Figure 1.7 (Example 9)

Solve Graphically Our goal is to predict the population in the year 2010. Reading from the graph in Figure 1.7b, we conclude that when x is 2010, y is approximately 6865.

Confirm Algebraically Evaluating Equation 1 for $x = 2010$ gives

$$\begin{aligned} y &= 79.957(2010) - 153848.716 \\ &\approx 6865. \end{aligned}$$

Interpret The linear regression equation suggests that the world population in the year 2010 will be about 6865 million, or approximately 53 million more than the Statistical Abstract prediction of 6812 million. ***Now try Exercise 45.***

Regression Analysis

Regression analysis has four steps:

1. Plot the data (scatter plot).
2. Find the regression equation. For a line, it has the form $y = mx + b$.
3. Superimpose the graph of the regression equation on the scatter plot to see the fit.
4. Use the regression equation to predict y-values for particular values of x.

Quick Review 1.1 *(For help, go to Section 1.1.)*

1. Find the value of y that corresponds to $x = 3$ in $y = -2 + 4(x - 3)$.

2. Find the value of x that corresponds to $y = 3$ in $y = 3 - 2(x + 1)$.

In Exercises 3 and 4, find the value of m that corresponds to the values of x and y.

3. $x = 5, \quad y = 2, \quad m = \dfrac{y - 3}{x - 4}$

4. $x = -1, \quad y = -3, \quad m = \dfrac{2 - y}{3 - x}$

In Exercises 5 and 6, determine whether the ordered pair is a solution to the equation.

5. $3x - 4y = 5$
 (a) $(2, 1/4)$ **(b)** $(3, -1)$

6. $y = -2x + 5$
 (a) $(-1, 7)$ **(b)** $(-2, 1)$

In Exercises 7 and 8, find the distance between the points.

7. $(1, 0), \quad (0, 1)$

8. $(2, 1), \quad (1, -1/3)$

In Exercises 9 and 10, solve for y in terms of x.

9. $4x - 3y = 7$

10. $-2x + 5y = -3$

Section 1.1 Exercises

In Exercises 1–4, find the coordinate increments from A to B.

1. $A(1, 2), \quad B(-1, -1)$

2. $A(-3, 2), \quad B(-1, -2)$

3. $A(-3, 1), \quad B(-8, 1)$

4. $A(0, 4), \quad B(0, -2)$

In Exercises 5–8, let L be the line determined by points A and B.
 (a) Plot A and B. **(b)** Find the slope of L.
 (c) Draw the graph of L.

5. $A(1, -2), \quad B(2, 1)$

6. $A(-2, -1), \quad B(1, -2)$

7. $A(2, 3), \quad B(-1, 3)$

8. $A(1, 2), \quad B(1, -3)$

In Exercise 9–12, write an equation for **(a)** the vertical line and **(b)** the horizontal line through the point P.

9. $P(3, 2)$

10. $P(-1, 4/3)$

11. $P(0, -\sqrt{2})$

12. $P(-\pi, 0)$

In Exercises 13–16, write the point-slope equation for the line through the point P with slope m.

13. $P(1, 1), \quad m = 1$

14. $P(-1, 1), \quad m = -1$

15. $P(0, 3), \quad m = 2$

16. $P(-4, 0), \quad m = -2$

In Exercises 17–20, write the slope-intercept equation for the line with slope m and y-intercept b.

17. $m = 3, \quad b = -2$

18. $m = -1, \quad b = 2$

19. $m = -1/2, \quad b = -3$

20. $m = 1/3, \quad b = -1$

In Exercises 21–24, write a general linear equation for the line through the two points.

21. $(0, 0), \quad (2, 3)$

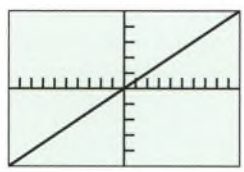
22. $(1, 1), \quad (2, 1)$

23. $(-2, 0), \quad (-2, -2)$

24. $(-2, 1), \quad (2, -2)$

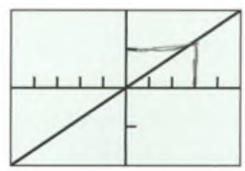
In Exercises 25 and 26, the line contains the origin and the point in the upper right corner of the grapher screen. Write an equation for the line.

25.



[–10, 10] by [–25, 25]

26.



[–5, 5] by [–2, 2]

In Exercises 27–30, find the **(a)** slope and **(b)** y-intercept, and **(c)** graph the line.

27. $3x + 4y = 12$

28. $x + y = 2$

29. $\dfrac{x}{3} + \dfrac{y}{4} = 1$

30. $y = 2x + 4$

In Exercises 31–34, write an equation for the line through P that is **(a)** parallel to L, and **(b)** perpendicular to L.

31. $P(0, 0), \quad L\colon y = -x + 2$

32. $P(-2, 2), \quad L\colon 2x + y = 4$

33. $P(-2, 4), \quad L\colon x = 5$

34. $P(-1, 1/2), \quad L\colon y = 3$

In Exercises 35 and 36, a table of values is given for the linear function $f(x) = mx + b$. Determine m and b.

35.

x	$f(x)$
1	2
3	9
5	16

36.

x	$f(x)$
2	−1
4	−4
6	−7

In Exercises 37 and 38, find the value of x or y for which the line through A and B has the given slope m.

37. $A(-2, 3)$, $B(4, y)$, $m = -2/3$

38. $A(-8, -2)$, $B(x, 2)$, $m = 2$

39. ***Revisiting Example 4*** Show that you get the same equation in Example 4 if you use the point (3, 4) to write the equation.

40. **Writing to Learn** ***x- and y-intercepts***

(a) Explain why c and d are the x-intercept and y-intercept, respectively, of the line

$$\frac{x}{c} + \frac{y}{d} = 1.$$

(b) How are the x-intercept and y-intercept related to c and d in the line

$$\frac{x}{c} + \frac{y}{d} = 2?$$

41. ***Parallel and Perpendicular Lines*** For what value of k are the two lines $2x + ky = 3$ and $x + y = 1$ **(a)** parallel? **(b)** perpendicular?

Group Activity In Exercises 42–44, work in groups of two or three to solve the problem.

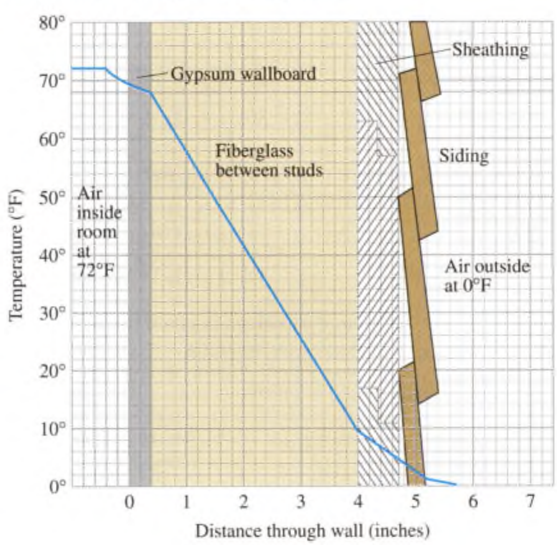
42. ***Insulation*** By measuring slopes in the figure below, find the temperature change in degrees per inch for the following materials.

(a) gypsum wallboard

(b) fiberglass insulation

(c) wood sheathing

(d) **Writing to Learn** Which of the materials in (a)–(c) is the best insulator? the poorest? Explain.



43. ***Pressure under Water*** The pressure p experienced by a diver under water is related to the diver's depth d by an equation of the form $p = kd + 1$ (k a constant). When $d = 0$ meters, the pressure is 1 atmosphere. The pressure at 100 meters is 10.94 atmospheres. Find the pressure at 50 meters.

44. ***Modeling Distance Traveled*** A car starts from point P at time $t = 0$ and travels at 45 mph.

(a) Write an expression $d(t)$ for the distance the car travels from P.

(b) Graph $y = d(t)$.

(c) What is the slope of the graph in (b)? What does it have to do with the car?

(d) **Writing to Learn** Create a scenario in which t could have negative values.

(e) **Writing to Learn** Create a scenario in which the y-intercept of $y = d(t)$ could be 30.

In Exercises 45 and 46, use linear regression analysis.

45. Table 1.2 shows the mean annual compensation of construction workers.

Table 1.2 Construction Workers' Average Annual Compensation

Year	Annual Total Compensation (dollars)
1999	42,598
2000	44,764
2001	47,822
2002	48,966

Source: U.S. Bureau of the Census, *Statistical Abstract of the United States, 2004–2005.*

(a) Find the linear regression equation for the data.

(b) Find the slope of the regression line. What does the slope represent?

(c) Superimpose the graph of the linear regression equation on a scatter plot of the data.

(d) Use the regression equation to predict the construction workers' average annual compensation in the year 2008.

46. Table 1.3 lists the ages and weights of nine girls.

Table 1.3 Girls' Ages and Weights

Age (months)	Weight (pounds)
19	22
21	23
24	25
27	28
29	31
31	28
34	32
38	34
43	39

(a) Find the linear regression equation for the data.

(b) Find the slope of the regression line. What does the slope represent?

(c) Superimpose the graph of the linear regression equation on a scatter plot of the data.

(d) Use the regression equation to predict the approximate weight of a 30-month-old girl.

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

47. True or False The slope of a vertical line is zero. Justify your answer.

48. True or False The slope of a line perpendicular to the line $y = mx + b$ is $1/m$. Justify your answer.

49. Multiple Choice Which of the following is an equation of the line through $(-3, 4)$ with slope $1/2$?

(A) $y - 4 = \frac{1}{2}(x + 3)$ **(B)** $y + 3 = \frac{1}{2}(x - 4)$

(C) $y - 4 = -2(x + 3)$ **(D)** $y - 4 = 2(x + 3)$

(E) $y + 3 = 2(x - 4)$

50. Multiple Choice Which of the following is an equation of the vertical line through $(-2, 4)$?

(A) $y = 4$ **(B)** $x = 2$ **(C)** $y = -4$

(D) $x = 0$ **(E)** $x = -2$

51. Multiple Choice Which of the following is the x-intercept of the line $y = 2x - 5$?

(A) $x = -5$ **(B)** $x = 5$ **(C)** $x = 0$

(D) $x = 5/2$ **(E)** $x = -5/2$

52. Multiple Choice Which of the following is an equation of the line through $(-2, -1)$ parallel to the line $y = -3x + 1$?

(A) $y = -3x + 5$ **(B)** $y = -3x - 7$ **(C)** $y = \frac{1}{3}x - \frac{1}{3}$

(D) $y = -3x + 1$ **(E)** $y = -3x - 4$

Extending the Ideas

53. The median price of existing single-family homes has increased consistently during the past few years. However, the data in Table 1.4 show that there have been differences in various parts of the country.

Table 1.4 Median Price of Single-Family Homes

Year	South (dollars)	West (dollars)
1999	145,900	173,700
2000	148,000	196,400
2001	155,400	213,600
2002	163,400	238,500
2003	168,100	260,900

Source: U.S. Bureau of the Census, *Statistical Abstract of the United States, 2004–2005.*

(a) Find the linear regression equation for home cost in the South.

(b) What does the slope of the regression line represent?

(c) Find the linear regression equation for home cost in the West.

(d) Where is the median price increasing more rapidly, in the South or the West?

54. *Fahrenheit versus Celsius* We found a relationship between Fahrenheit temperature and Celsius temperature in Example 8.

(a) Is there a temperature at which a Fahrenheit thermometer and a Celsius thermometer give the same reading? If so, what is it?

(b) Writing to Learn Graph $y_1 = (9/5)x + 32$, $y_2 = (5/9)(x - 32)$, and $y_3 = x$ in the same viewing window. Explain how this figure is related to the question in part (a).

55. *Parallelogram* Three different parallelograms have vertices at $(-1, 1)$, $(2, 0)$, and $(2, 3)$. Draw the three and give the coordinates of the missing vertices.

56. *Parallelogram* Show that if the midpoints of consecutive sides of any quadrilateral are connected, the result is a parallelogram.

57. *Tangent Line* Consider the circle of radius 5 centered at $(0, 0)$. Find an equation of the line tangent to the circle at the point $(3, 4)$.

58. Group Activity *Distance From a Point to a Line* This activity investigates how to find the distance from a point $P(a, b)$ to a line $L: Ax + By = C$.

(a) Write an equation for the line M through P perpendicular to L.

(b) Find the coordinates of the point Q in which M and L intersect.

(c) Find the distance from P to Q.

1.2 Functions and Graphs

What you'll learn about

- Functions
- Domains and Ranges
- Viewing and Interpreting Graphs
- Even Functions and Odd Functions—Symmetry
- Functions Defined in Pieces
- Absolute Value Function
- Composite Functions

. . . and why

Functions and graphs form the basis for understanding mathematics and applications.

Functions

The values of one variable often depend on the values for another:

- The temperature at which water boils depends on elevation (the boiling point drops as you go up).
- The amount by which your savings will grow in a year depends on the interest rate offered by the bank.
- The area of a circle depends on the circle's radius.

In each of these examples, the value of one variable quantity depends on the value of another. For example, the boiling temperature of water, b, depends on the elevation, e; the amount of interest, I, depends on the interest rate, r. We call b and I **dependent variables** because they are determined by the values of the variables e and r on which they depend. The variables e and r are **independent variables.**

A rule that assigns to each element in one set a unique element in another set is called a *function.* The sets may be sets of any kind and do not have to be the same. A function is like a machine that assigns a unique output to every allowable input. The inputs make up the **domain** of the function; the outputs make up the **range** (Figure 1.8).

Figure 1.8 A "machine" diagram for a function.

DEFINITION Function

A **function** from a set D to a set R is a rule that assigns a unique element in R to each element in D.

In this definition, D is the domain of the function and R is a set *containing* the range (Figure 1.9).

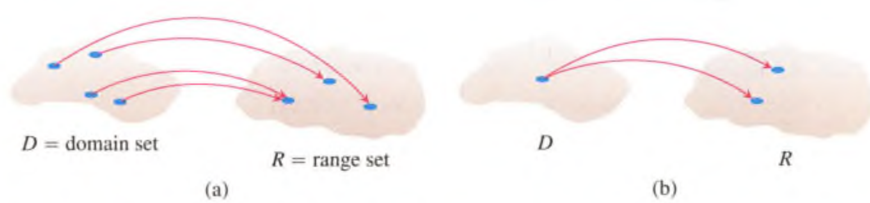


Figure 1.9 (a) A function from a set D to a set R. (b) *Not* a function. The assignment is not unique.

Leonhard Euler

(1707–1783)

Leonhard Euler, the dominant mathematical figure of his century and the most prolific mathematician ever, was also an astronomer, physicist, botanist, and chemist, and an expert in oriental languages. His work was the first to give the function concept the prominence that it has in mathematics today. Euler's collected books and papers fill 72 volumes. This does not count his enormous correspondence to approximately 300 addresses. His introductory algebra text, written originally in German (Euler was Swiss), is still available in English translation.

Euler invented a symbolic way to say "y is a function of x":

$$y = f(x),$$

which we read as "y equals f of x." This notation enables us to give different functions different names by changing the letters we use. To say that the boiling point of water is a function of elevation, we can write $b = f(e)$. To say that the area of a circle is a function of the circle's radius, we can write $A = A(r)$, giving the function the same name as the dependent variable.

The notation $y = f(x)$ gives a way to denote specific values of a function. The value of f at a can be written as $f(a)$, read "f of a."

EXAMPLE 1 The Circle-Area Function

Write a formula that expresses the area of a circle as a function of its radius. Use the formula to find the area of a circle of radius 2 in.

SOLUTION

If the radius of the circle is r, then the area $A(r)$ of the circle can be expressed as $A(r) = \pi r^2$. The area of a circle of radius 2 can be found by evaluating the function $A(r)$ at $r = 2$.

$$A(2) = \pi(2)^2 = 4\pi$$

The area of a circle of radius 2 is 4π in^2. *Now try Exercise 3.*

Domains and Ranges

In Example 1, the domain of the function is restricted by context: the independent variable is a radius and must be positive. When we define a function $y = f(x)$ with a formula and the domain is not stated explicitly or restricted by context, the domain is assumed to be the largest set of x-values for which the formula gives real y-values—the so-called **natural domain.** If we want to restrict the domain, we must say so. The domain of $y = x^2$ is understood to be the entire set of real numbers. We must write "$y = x^2, x > 0$" if we want to restrict the function to positive values of x.

The domains and ranges of many real-valued functions of a real variable are intervals or combinations of intervals. The intervals may be open, closed, or half-open (Figures 1.10 and 1.11) and finite or infinite (Figure 1.12).

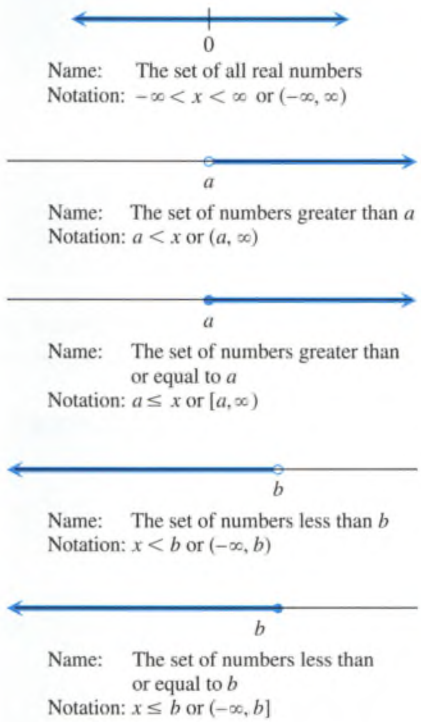


Figure 1.12 Infinite intervals—rays on the number line and the number line itself. The symbol ∞ (infinity) is used merely for convenience; it does not mean there is a number ∞.

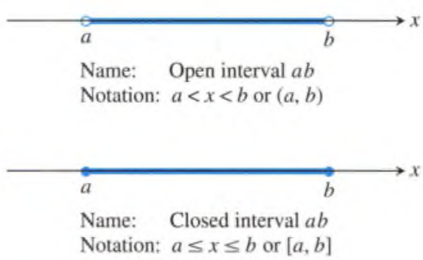


Figure 1.10 Open and closed finite intervals.

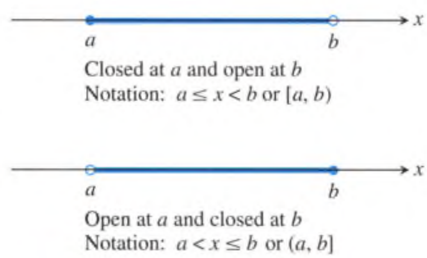


Figure 1.11 Half-open finite intervals.

The endpoints of an interval make up the interval's **boundary** and are called **boundary points.** The remaining points make up the interval's **interior** and are called **interior points. Closed intervals** contain their boundary points. **Open intervals** contain no boundary points. Every point of an open interval is an interior point of the interval.

Viewing and Interpreting Graphs

The points (x, y) in the plane whose coordinates are the input-output pairs of a function $y = f(x)$ make up the function's **graph.** The graph of the function $y = x + 2$, for example, is the set of points with coordinates (x, y) for which y equals $x + 2$.

EXAMPLE 2 Identifying Domain and Range of a Function

Identify the domain and range, and then sketch a graph of the function.

(a) $y = \dfrac{1}{x}$ **(b)** $y = \sqrt{x}$

SOLUTION

(a) The formula gives a real y-value for every real x-value except $x = 0$. (*We cannot divide any number by* 0.) The domain is $(-\infty, 0) \cup (0, \infty)$. The value y takes on every real number except $y = 0$. ($y = c \neq 0$ if $x = 1/c$) The range is also $(-\infty, 0) \cup (0, \infty)$. A sketch is shown in Figure 1.13a.

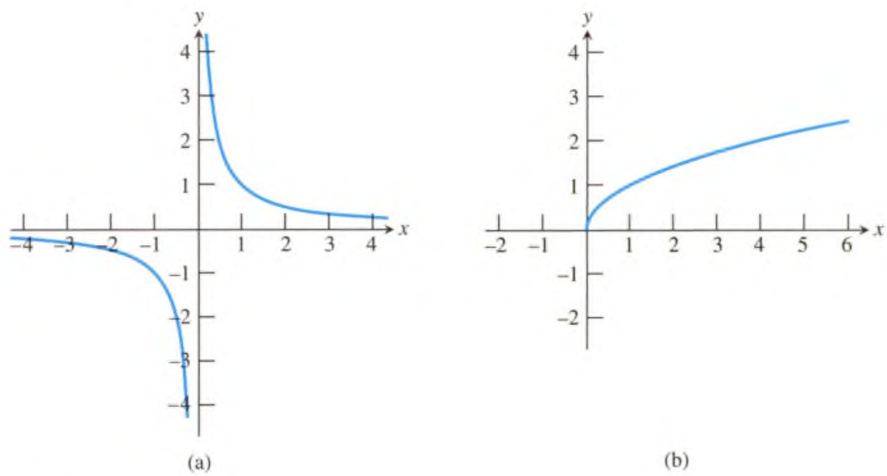


Figure 1.13 A sketch of the graph of (a) $y = 1/x$ and (b) $y = \sqrt{x}$. (Example 2)

(b) The formula gives a real number only when x is positive or zero. The domain is $[0, \infty)$. Because $\sqrt{x}$ denotes the principal square root of x, y is greater than or equal to zero. The range is also $[0, \infty)$. A sketch is shown in Figure 1.13b.

Now try Exercise 9.

Graphing with pencil and paper requires that you develop graph *drawing* skills. Graphing with a grapher (graphing calculator) requires that you develop graph *viewing* skills.

Power Function

Any function that can be written in the form $f(x) = kx^a$, where k and a are nonzero constants, is a **power function**.

Graph Viewing Skills

1. Recognize that the graph is reasonable.
2. See all the important characteristics of the graph.
3. Interpret those characteristics.
4. Recognize grapher failure.

Being able to recognize that a graph is reasonable comes with experience. You need to know the basic functions, their graphs, and how changes in their equations affect the graphs.

Grapher failure occurs when the graph produced by a grapher is less than precise—or even incorrect—usually due to the limitations of the screen resolution of the grapher.

EXAMPLE 3 Identifying Domain and Range of a Function

Use a grapher to identify the domain and range, and then draw a graph of the function.

(a) $y = \sqrt{4 - x^2}$ **(b)** $y = x^{2/3}$

SOLUTION

(a) Figure 1.14a shows a graph of the function for $-4.7 \le x \le 4.7$ and $-3.1 \le y \le 3.1$, that is, the viewing window $[-4.7, 4.7]$ by $[-3.1, 3.1]$, with x-scale = y-scale = 1. The graph appears to be the upper half of a circle. The domain appears to be $[-2, 2]$. This observation is correct because we must have $4 - x^2 \ge 0$, or equivalently, $-2 \le x \le 2$. The range appears to be $[0, 2]$, which can also be verified algebraically.

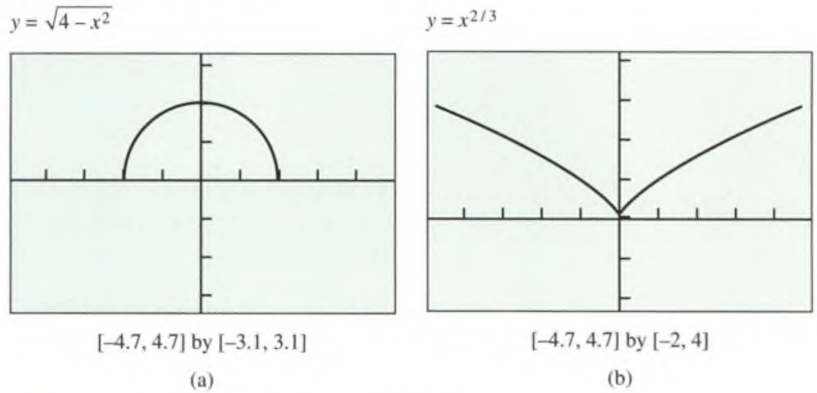


Figure 1.14 The graph of (a) $y = \sqrt{4 - x^2}$ and (b) $y = x^{2/3}$. (Example 3)

(b) Figure 1.14b shows a graph of the function in the viewing window $[-4.7, 4.7]$ by $[-2, 4]$, with x-scale = y-scale = 1. The domain appears to be $(-\infty, \infty)$, which we can verify by observing that $x^{2/3} = (\sqrt[3]{x})^2$. Also the range is $[0, \infty)$ by the same observation. ***Now try Exercise 15.***

Graphing $y = x^{2/3}$—Possible Grapher Failure

On some graphing calculators you need to enter this function as $y = (x^2)^{1/3}$ or $y = (x^{1/3})^2$ to obtain a correct graph. Try graphing this function on your grapher.

Even Functions and Odd Functions—Symmetry

The graphs of *even* and *odd* functions have important symmetry properties.

DEFINITIONS Even Function, Odd Function

A function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$,

odd function of x if $f(-x) = -f(x)$,

for every x in the function's domain.

The names even and odd come from powers of x. If y is an even power of x, as in $y = x^2$ or $y = x^4$, it is an even function of x (because $(-x)^2 = x^2$ and $(-x)^4 = x^4$). If y is an odd power of x, as in $y = x$ or $y = x^3$, it is an odd function of x (because $(-x)^1 = -x$ and $(-x)^3 = -x^3$).

The graph of an even function is **symmetric about the y-axis.** Since $f(-x) = f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, y)$ lies on the graph (Figure 1.15a).

The graph of an odd function is **symmetric about the origin.** Since $f(-x) = -f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, -y)$ lies on the graph (Figure 1.15b).

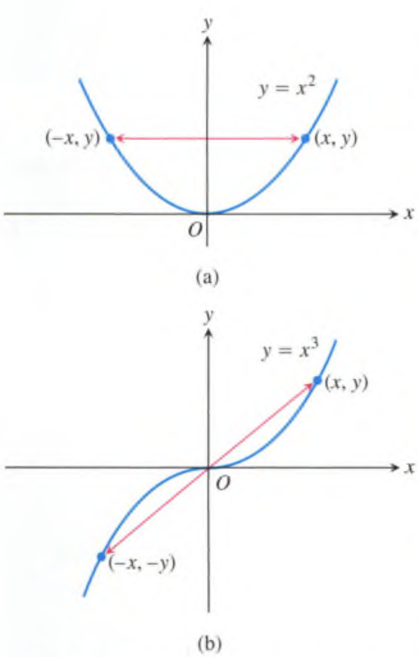


Figure 1.15 (a) The graph of $y = x^2$ (an even function) is symmetric about the y-axis. (b) The graph of $y = x^3$ (an odd function) is symmetric about the origin.

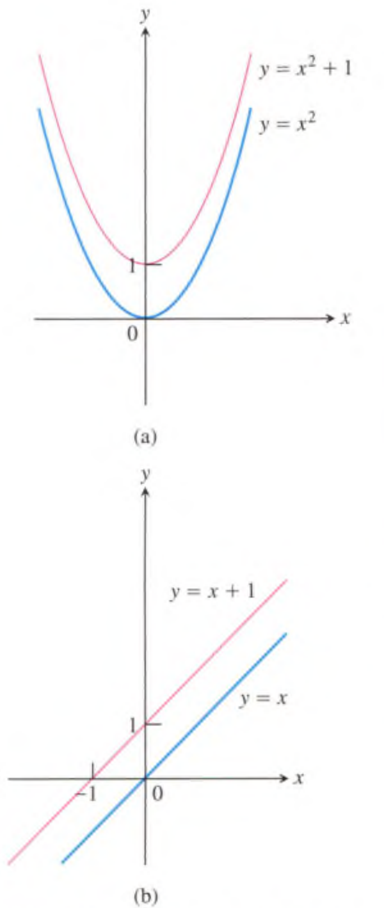


Figure 1.16 (a) When we add the constant term 1 to the function $y = x^2$, the resulting function $y = x^2 + 1$ is still even and its graph is still symmetric about the y-axis. (b) When we add the constant term 1 to the function $y = x$, the resulting function $y = x + 1$ is no longer odd. The symmetry about the origin is lost. (Example 4)

Equivalently, a graph is symmetric about the origin if a rotation of 180° about the origin leaves the graph unchanged.

EXAMPLE 4 Recognizing Even and Odd Functions

$f(x) = x^2$	Even function: $(-x)^2 = x^2$ for all x; symmetry about y-axis.
$f(x) = x^2 + 1$	Even function: $(-x)^2 + 1 = x^2 + 1$ for all x; symmetry about y-axis (Figure 1.16a).
$f(x) = x$	Odd function: $(-x) = -x$ for all x; symmetry about the origin.
$f(x) = x + 1$	Not odd: $f(-x) = -x + 1$, but $-f(x) = -x - 1$. The two are not equal.
	Not even: $(-x) + 1 \neq x + 1$ for all $x \neq 0$ (Figure 1.16b).

Now try Exercises 21 and 23.

It is useful in graphing to recognize even and odd functions. Once we know the graph of either type of function on one side of the y-axis, we know its graph on both sides.

Functions Defined in Pieces

While some functions are defined by single formulas, others are defined by applying different formulas to different parts of their domains.

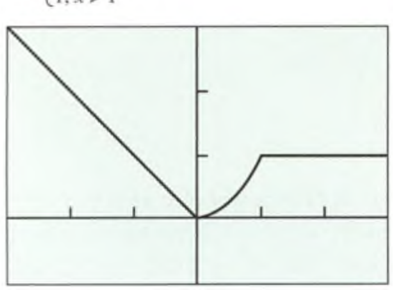
EXAMPLE 5 Graphing Piecewise-Defined Functions

$$\text{Graph } y = f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \le x \le 1 \\ 1, & x > 1. \end{cases}$$

SOLUTION

The values of f are given by three separate formulas: $y = -x$ when $x < 0$, $y = x^2$ when $0 \le x \le 1$, and $y = 1$ when $x > 1$. However, the function is *just one function,* whose domain is the entire set of real numbers (Figure 1.17). ***Now try Exercise 33.***

$$y = \begin{cases} -x, x < 0 \\ x^2, 0 \le x \le 1 \\ 1, x > 1 \end{cases}$$



[–3, 3] by [–1, 3]

Figure 1.17 The graph of a piecewise defined function. (Example 5).

EXAMPLE 6 Writing Formulas for Piecewise Functions

Write a formula for the function $y = f(x)$ whose graph consists of the two line segments in Figure 1.18.

SOLUTION

We find formulas for the segments from (0, 0) to (1, 1) and from (1, 0) to (2, 1) and piece them together in the manner of Example 5.

Segment from (0, 0) to (1, 1) The line through (0, 0) and (1, 1) has slope $m = (1 - 0)/(1 - 0) = 1$ and y-intercept $b = 0$. Its slope-intercept equation is $y = x$. The segment from (0, 0) to (1, 1) that includes the point (0, 0) but not the point (1, 1) is the graph of the function $y = x$ restricted to the half-open interval $0 \le x < 1$, namely,

$$y = x, \quad 0 \le x < 1.$$

continued

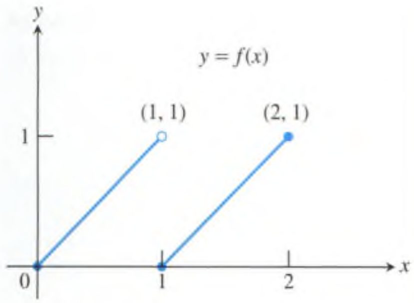


Figure 1.18 The segment on the left contains (0, 0) but not (1, 1). The segment on the right contains both of its endpoints. (Example 6)

Segment from (1, 0) to (2, 1) The line through (1, 0) and (2, 1) has slope $m = (1 - 0)/(2 - 1) = 1$ and passes through the point (1, 0). The corresponding point-slope equation for the line is

$$y = 1(x - 1) + 0, \quad \text{or} \quad y = x - 1.$$

The segment from (1, 0) to (2, 1) that includes both endpoints is the graph of $y = x - 1$ restricted to the closed interval $1 \le x \le 2$, namely,

$$y = x - 1, \quad 1 \le x \le 2.$$

Piecewise Formula Combining the formulas for the two pieces of the graph, we obtain

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ x - 1, & 1 \le x \le 2. \end{cases}$$

Now try Exercise 43.

Absolute Value Function

The **absolute value function** $y = |x|$ is defined piecewise by the formula

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \ge 0. \end{cases}$$

The function is even, and its graph (Figure 1.19) is symmetric about the y-axis.

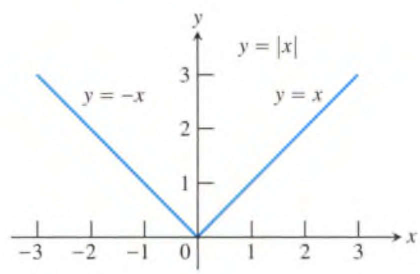
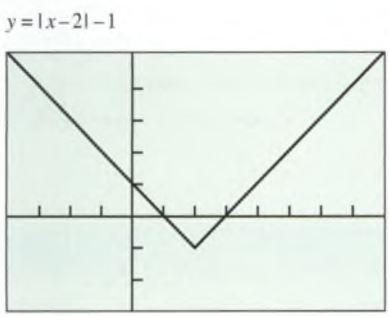


Figure 1.19 The absolute value function has domain $(-\infty, \infty)$ and range $[0, \infty)$.



[–4, 8] by [–3, 5]

Figure 1.20 The lowest point of the graph of $f(x) = |x - 2| - 1$ is $(2, -1)$. (Example 7)

EXAMPLE 7 Using Transformations

Draw the graph of $f(x) = |x - 2| - 1$. Then find the domain and range.

SOLUTION

The graph of f is the graph of the absolute value function shifted 2 units horizontally to the right and 1 unit vertically downward (Figure 1.20). The domain of f is $(-\infty, \infty)$ and the range is $[-1, \infty)$.

Now try Exercise 49.

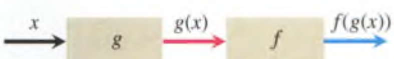


Figure 1.21 Two functions can be composed when a portion of the range of the first lies in the domain of the second.

Composite Functions

Suppose that some of the outputs of a function g can be used as inputs of a function f. We can then link g and f to form a new function whose inputs x are inputs of g and whose outputs are the numbers $f(g(x))$, as in Figure 1.21. We say that the function $f(g(x))$ (read

"f of g of x") is the **composite of g and f.** It is made by *composing* g and f in the order of first g, then f. The usual "stand-alone" notation for this composite is $f \circ g$, which is read as "f of g." Thus, the value of $f \circ g$ at x is $(f \circ g)(x) = f(g(x))$.

EXAMPLE 8 Composing Functions

Find a formula for $f(g(x))$ if $g(x) = x^2$ and $f(x) = x - 7$. Then find $f(g(2))$.

SOLUTION

To find $f(g(x))$, we replace x in the formula $f(x) = x - 7$ by the expression given for $g(x)$.

$$f(x) = x - 7$$
$$f(g(x)) = g(x) - 7 = x^2 - 7$$

We then find the value of $f(g(2))$ by substituting 2 for x.

$$f(g(2)) = (2)^2 - 7 = -3$$

Now try Exercise 51.

EXPLORATION 1 Composing Functions

Some graphers allow a function such as y_1 to be used as the independent variable of another function. With such a grapher, we can compose functions.

1. Enter the functions $y_1 = f(x) = 4 - x^2$, $y_2 = g(x) = \sqrt{x}$, $y_3 = y_2(y_1(x))$, and $y_4 = y_1(y_2(x))$. Which of y_3 and y_4 corresponds to $f \circ g$? to $g \circ f$?
2. Graph y_1, y_2, and y_3 and make conjectures about the domain and range of y_3.
3. Graph y_1, y_2, and y_4 and make conjectures about the domain and range of y_4.
4. Confirm your conjectures algebraically by finding formulas for y_3 and y_4.

Quick Review 1.2 *(For help, go to Appendix A1 and Section 1.2.)*

In Exercises 1–6, solve for x.

1. $3x - 1 \le 5x + 3$ **2.** $x(x - 2) > 0$

3. $|x - 3| \le 4$ **4.** $|x - 2| \ge 5$

5. $x^2 < 16$ **6.** $9 - x^2 \ge 0$

In Exercises 7 and 8, describe how the graph of f can be transformed to the graph of g.

7. $f(x) = x^2$, $g(x) = (x + 2)^2 - 3$

8. $f(x) = |x|$, $g(x) = |x - 5| + 2$

In Exercises 9–12, find all real solutions to the equations.

9. $f(x) = x^2 - 5$
 (a) $f(x) = 4$ **(b)** $f(x) = -6$

10. $f(x) = 1/x$
 (a) $f(x) = -5$ **(b)** $f(x) = 0$

11. $f(x) = \sqrt{x + 7}$
 (a) $f(x) = 4$ **(b)** $f(x) = 1$

12. $f(x) = \sqrt[3]{x - 1}$
 (a) $f(x) = -2$ **(b)** $f(x) = 3$

Section 1.2 Exercises

In Exercises 1–4, **(a)** write a formula for the function and **(b)** use the formula to find the indicated value of the function.

1. the area A of a circle as a function of its diameter d; the area of a circle of diameter 4 in.

2. the height h of an equilateral triangle as a function of its side length s; the height of an equilateral triangle of side length 3 m

3. the surface area S of a cube as a function of the length of the cube's edge e; the surface area of a cube of edge length 5 ft

4. the volume V of a sphere as a function of the sphere's radius r; the volume of a sphere of radius 3 cm

In Exercises 5–12, **(a)** identify the domain and range and **(b)** sketch the graph of the function.

5. $y = 4 - x^2$ **6.** $y = x^2 - 9$

7. $y = 2 + \sqrt{x - 1}$ **8.** $y = -\sqrt{-x}$

9. $y = \dfrac{1}{x - 2}$ **10.** $y = \sqrt[4]{-x}$

11. $y = 1 + \dfrac{1}{x}$ **12.** $y = 1 + \dfrac{1}{x^2}$

In Exercises 13–20, use a grapher to **(a)** identify the domain and range and **(b)** draw the graph of the function.

13. $y = \sqrt[3]{x}$ **14.** $y = 2\sqrt{3 - x}$

15. $y = \sqrt[3]{1 - x^2}$ **16.** $y = \sqrt{9 - x^2}$

17. $y = x^{2/5}$ **18.** $y = x^{3/2}$

19. $y = \sqrt[3]{x - 3}$ **20.** $y = \dfrac{1}{\sqrt{4 - x^2}}$

In Exercises 21–30, determine whether the function is even, odd, or neither. Try to answer without writing anything (except the answer).

21. $y = x^4$ **22.** $y = x + x^2$

23. $y = x + 2$ **24.** $y = x^2 - 3$

25. $y = \sqrt{x^2 + 2}$ **26.** $y = x + x^3$

27. $y = \dfrac{x^3}{x^2 - 1}$ **28.** $y = \sqrt[3]{2 - x}$

29. $y = \dfrac{1}{x - 1}$ **30.** $y = \dfrac{1}{x^2 - 1}$

In Exercises 31–34, graph the piecewise-defined functions.

31. $f(x) = \begin{cases} 3 - x, & x \le 1 \\ 2x, & 1 < x \end{cases}$ **32.** $f(x) = \begin{cases} 1, & x < 0 \\ \sqrt{x}, & x \ge 0 \end{cases}$

33. $f(x) = \begin{cases} 4 - x^2, & x < 1 \\ (3/2)x + 3/2, & 1 \le x \le 3 \\ x + 3, & x > 3 \end{cases}$

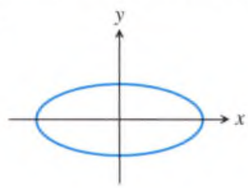
34. $f(x) = \begin{cases} x^2, & x < 0 \\ x^3, & 0 \le x \le 1 \\ 2x - 1, & x > 1 \end{cases}$

35. **Writing to Learn** The *vertical line test* to determine whether a curve is the graph of a function states: If every vertical line in the xy-plane intersects a given curve in at most one point, then the curve is the graph of a function. Explain why this is true.

36. **Writing to Learn** For a curve to be *symmetric about the x-axis*, the point (x, y) must lie on the curve if and only if the point $(x, -y)$ lies on the curve. Explain why a curve that is symmetric about the x-axis is not the graph of a function, unless the function is $y = 0$.

In Exercises 37–40, use the vertical line test (see Exercise 35) to determine whether the curve is the graph of a function.

37.



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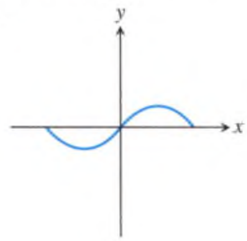

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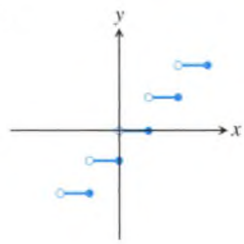

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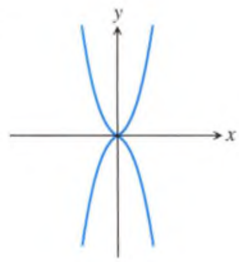
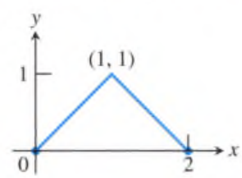


In Exercises 41–48, write a piecewise formula for the function.

41.



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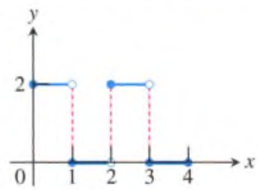

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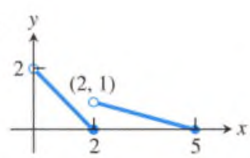

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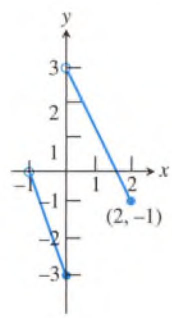

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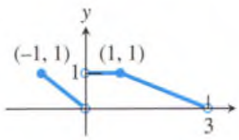

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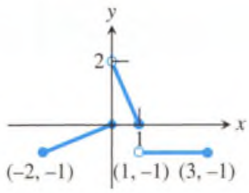

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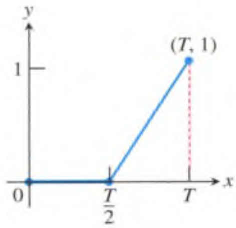

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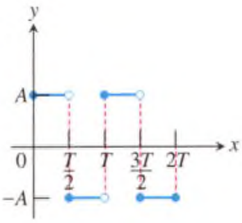


In Exercises 49 and 50, **(a)** draw the graph of the function. Then find its **(b)** domain and **(c)** range.

49. $f(x) = -|3 - x| + 2$ **50.** $f(x) = 2|x + 4| - 3$

In Exercises 51 and 52, find

(a) $f(g(x))$ **(b)** $g(f(x))$ **(c)** $f(g(0))$

(d) $g(f(0))$ **(e)** $g(g(-2))$ **(f)** $f(f(x))$

51. $f(x) = x + 5,\quad g(x) = x^2 - 3$

52. $f(x) = x + 1,\quad g(x) = x - 1$

53. Copy and complete the following table.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
(a)	?	$\sqrt{x-5}$	$\sqrt{x^2-5}$
(b)	?	$1+1/x$	x
(c)	$1/x$	?	x
(d)	$\sqrt{x}$	?	$\|x\|, x \geq 0$

54. ***Broadway Season Statistics*** Table 1.5 shows the gross revenue for the Broadway season in millions of dollars for several years.

Table 1.5 Broadway Season Revenue

Year	Amount ($ millions)
1997	558
1998	588
1999	603
2000	666
2001	643
2002	721
2003	771

Source: The League of American Theatres and Producers, Inc., New York, NY, as reported in *The World Almanac and Book of Facts, 2005.*

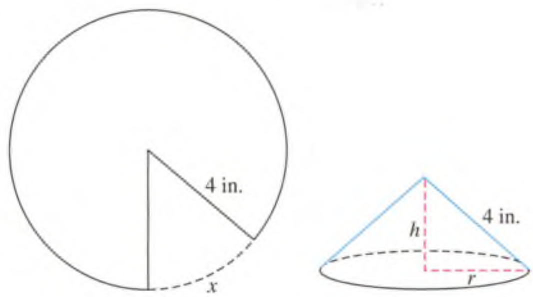
(a) Find the quadratic regression for the data in Table 1.5. Let $x = 0$ represent 1990, $x = 1$ represent 1991, and so forth.

(b) Superimpose the graph of the quadratic regression equation on a scatter plot of the data.

(c) Use the quadratic regression to predict the amount of revenue in 2008.

(d) Now find the linear regression for the data and use it to predict the amount of revenue in 2008.

55. ***The Cone Problem*** Begin with a circular piece of paper with a 4-in. radius as shown in (a). Cut out a sector with an arc length of x. Join the two edges of the remaining portion to form a cone with radius r and height h, as shown in (b).



(a) (b)

(a) Explain why the circumference of the base of the cone is $8\pi - x$.

(b) Express the radius r as a function of x.

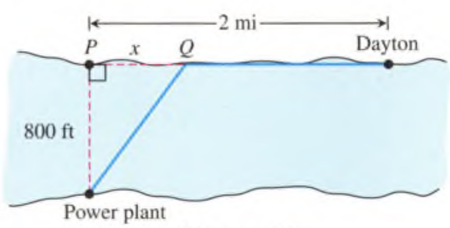
(c) Express the height h as a function of x.

(d) Express the volume V of the cone as a function of x.

56. ***Industrial Costs*** Dayton Power and Light, Inc., has a power plant on the Miami River where the river is 800 ft wide. To lay a new cable from the plant to a location in the city 2 mi downstream on the opposite side costs $180 per foot across the river and $100 per foot along the land.

(a) Suppose that the cable goes from the plant to a point Q on the opposite side that is x ft from the point P directly opposite the plant. Write a function $C(x)$ that gives the cost of laying the cable in terms of the distance x.

(b) Generate a table of values to determine if the least expensive location for point Q is less than 2000 ft or greater than 2000 ft from point P.



(Not to scale)

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

57. True or False The function $f(x) = x^4 + x^2 + x$ is an even function. Justify your answer.

58. True or False The function $f(x) = x^{-3}$ is an odd function. Justify your answer.

59. Multiple Choice Which of the following gives the domain of $f(x) = \dfrac{x}{\sqrt{9 - x^2}}$?

(A) $x \neq \pm 3$ **(B)** $(-3, 3)$ **(C)** $[-3, 3]$
(D) $(-\infty, -3) \cup (3, \infty)$ **(E)** $(3, \infty)$

60. Multiple Choice Which of the following gives the range of $f(x) = 1 + \dfrac{1}{x - 1}$?

(A) $(-\infty, 1) \cup (1, \infty)$ **(B)** $x \neq 1$ **(C)** all real numbers
(D) $(-\infty, 0) \cup (0, \infty)$ **(E)** $x \neq 0$

61. Multiple Choice If $f(x) = 2x - 1$ and $g(x) = x + 3$, which of the following gives $(f \circ g)(2)$?

(A) 2 **(B)** 6 **(C)** 7 **(D)** 9 **(E)** 10

62. Multiple Choice The length L of a rectangle is twice as long as its width W. Which of the following gives the area A of the rectangle as a function of its width?

(A) $A(W) = 3W$ **(B)** $A(W) = \dfrac{1}{2}W^2$ **(C)** $A(W) = 2W^2$
(D) $A(W) = W^2 + 2W$ **(E)** $A(W) = W^2 - 2W$

Explorations

In Exercises 63–66, **(a)** graph $f \circ g$ and $g \circ f$ and make a conjecture about the domain and range of each function. **(b)** Then confirm your conjectures by finding formulas for $f \circ g$ and $g \circ f$.

63. $f(x) = x - 7, \quad g(x) = \sqrt{x}$

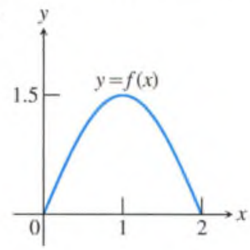
64. $f(x) = 1 - x^2, \quad g(x) = \sqrt{x}$

65. $f(x) = x^2 - 3, \quad g(x) = \sqrt{x + 2}$

66. $f(x) = \dfrac{2x - 1}{x + 3}, \quad g(x) = \dfrac{3x + 1}{2 - x}$

Group Activity In Exercises 67–70, a portion of the graph of a function defined on $[-2, 2]$ is shown. Complete each graph assuming that the graph is **(a)** even, **(b)** odd.

67.



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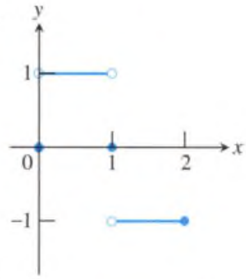

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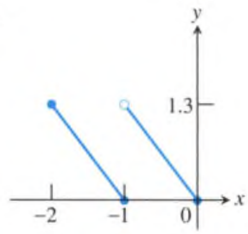

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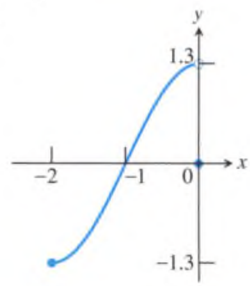


Extending the Ideas

71. Enter $y_1 = \sqrt{x}$, $y_2 = \sqrt{1 - x}$ and $y_3 = y_1 + y_2$ on your grapher.

(a) Graph y_3 in $[-3, 3]$ by $[-1, 3]$.

(b) Compare the domain of the graph of y_3 with the domains of the graphs of y_1 and y_2.

(c) Replace y_3 by

$$y_1 - y_2, \quad y_2 - y_1, \quad y_1 \cdot y_2, \quad y_1/y_2, \quad \text{and} \quad y_2/y_1,$$

in turn, and repeat the comparison of part (b).

(d) Based on your observations in (b) and (c), what would you conjecture about the domains of sums, differences, products, and quotients of functions?

72. ***Even and Odd Functions***

(a) Must the product of two even functions always be even? Give reasons for your answer.

(b) Can anything be said about the product of two odd functions? Give reasons for your answer.

1.3 Exponential Functions

What you'll learn about

- Exponential Growth
- Exponential Decay
- Applications
- The Number e

. . . and why

Exponential functions model many growth patterns.

Exponential Growth

Table 1.6 shows the growth of \$100 invested in 1996 at an interest rate of 5.5%, compounded annually.

Table 1.6 Savings Account Growth

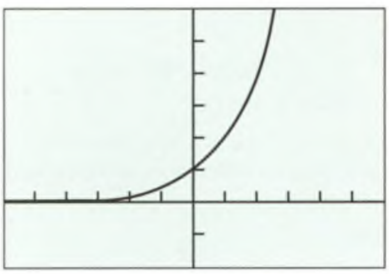
Year	Amount (dollars)	Increase (dollars)
1996	100	
1997	$100(1.055) = 105.50$	5.50
1998	$100(1.055)^2 = 111.30$	5.80
1999	$100(1.055)^3 = 117.42$	6.12
2000	$100(1.055)^4 = 123.88$	6.46

After the first year, the value of the account is always 1.055 times its value in the previous year. After n years, the value is $y = 100 \cdot (1.055)^n$.

Compound interest provides an example of *exponential growth* and is modeled by a function of the form $y = P \cdot a^x$, where P is the initial investment and a is equal to 1 plus the interest rate expressed as a decimal.

The equation $y = P \cdot a^x$, $a > 0$, $a \neq 1$, identifies a family of functions called *exponential functions.* Notice that the ratio of consecutive amounts in Table 1.6 is always the same: $111.30/105.30 = 117.42/111.30 = 123.88/117.42 \approx 1.055$. This fact is an important feature of exponential curves that has widespread application, as we will see.

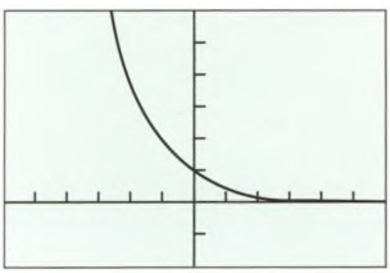
$y = 2^x$



[–6, 6] by [–2, 6]

(a)

$y = 2^{-x}$



[–6, 6] by [–2, 6]

(b)

Figure 1.22 A graph of (a) $y = 2^x$ and (b) $y = 2^{-x}$.

EXPLORATION 1 Exponential Functions

1. Graph the function $y = a^x$ for $a = 2, 3, 5$, in a $[-5, 5]$ by $[-2, 5]$ viewing window.
2. For what values of x is it true that $2^x < 3^x < 5^x$?
3. For what values of x is it true that $2^x > 3^x > 5^x$?
4. For what values of x is it true that $2^x = 3^x = 5^x$?
5. Graph the function $y = (1/a)^x = a^{-x}$ for $a = 2, 3, 5$.
6. Repeat parts 2–4 for the functions in part 5.

DEFINITION Exponential Function

Let a be a positive real number other than 1. The function

$$f(x) = a^x$$

is the **exponential function with base a.**

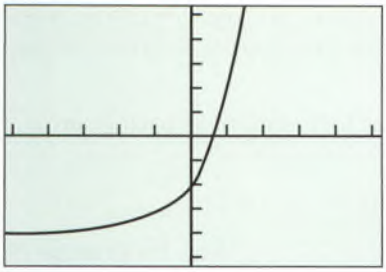
The domain of $f(x) = a^x$ is $(-\infty, \infty)$ and the range is $(0, \infty)$. If $a > 1$, the graph of f looks like the graph of $y = 2^x$ in Figure 1.22a. If $0 < a < 1$, the graph of f looks like the graph of $y = 2^{-x}$ in Figure 1.22b.

EXAMPLE 1 Graphing an Exponential Function

Graph the function $y = 2(3^x) - 4$. State its domain and range.

continued

$y = 2(3^x) - 4$



[–5, 5] by [–5, 5]

Figure 1.23 The graph of $y = 2(3^x) - 4$. (Example 1)

SOLUTION

Figure 1.23 shows the graph of the function y. It appears that the domain is $(-\infty, \infty)$. The range is $(-4, \infty)$ because $2(3^x) > 0$ for all x. *Now try Exercise 1.*

EXAMPLE 2 Finding Zeros

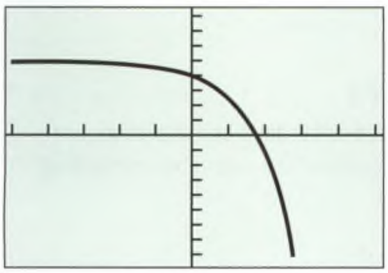
Find the zeros of $f(x) = 5 - 2.5^x$ graphically.

SOLUTION

Figure 1.24a suggests that f has a zero between $x = 1$ and $x = 2$, closer to 2. We can use our grapher to find that the zero is approximately 1.756 (Figure 1.24b).

Now try Exercise 9.

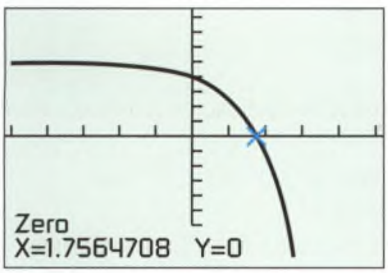
$y = 5 - 2.5^x$



[–5, 5] by [–8, 8]

(a)

$y = 5 - 2.5^x$



[–5, 5] by [–8, 8]

(b)

Figure 1.24 (a) A graph of $f(x) = 5 - 2.5^x$. (b) Showing the use of the ZERO feature to approximate the zero of f. (Example 2)

Exponential functions obey the rules for exponents.

Rules for Exponents

If $a > 0$ and $b > 0$, the following hold for all real numbers x and y.

1. $a^x \cdot a^y = a^{x+y}$
2. $\dfrac{a^x}{a^y} = a^{x-y}$
3. $(a^x)^y = (a^y)^x = a^{xy}$
4. $a^x \cdot b^x = (ab)^x$
5. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

In Table 1.6 we observed that the ratios of the amounts in consecutive years were always the same, namely the interest rate. Population growth can sometimes be modeled with an exponential function, as we see in Table 1.7 and Example 3.

Table 1.7 gives the United States population for several recent years. In this table we have divided the population in one year by the population in the previous year to get an idea of how the population is growing. These ratios are given in the third column.

Table 1.7 United States Population

Year	Population (millions)	Ratio
1998	276.1	
		279.3/276.1 ≈ 1.0116
1999	279.3	
		282.4/279.3 ≈ 1.0111
2000	282.4	
		285.3/282.4 ≈ 1.0102
2001	285.3	
		288.2/285.3 ≈ 1.0102
2002	288.2	
		291.0/288.2 ≈ 1.0097
2003	291.0	

Source: Statistical Abstract of the United States, 2004–2005.

EXAMPLE 3 Predicting United States Population

Use the data in Table 1.7 and an exponential model to predict the population of the United States in the year 2010.

continued

SOLUTION

Based on the third column of Table 1.7, we might be willing to conjecture that the population of the United States in any year is about 1.01 times the population in the previous year.

If we start with the population in 1998, then according to the model the population (in millions) in 2010 would be about

$$276.1(1.01)^{12} \approx 311.1,$$

or about 311.1 million people. *Now try Exercise 19.*

Exponential Decay

Exponential functions can also model phenomena that produce a decrease over time, such as happens with radioactive decay. The **half-life** of a radioactive substance is the amount of time it takes for half of the substance to change from its original radioactive state to a nonradioactive state by emitting energy in the form of radiation.

EXAMPLE 4 Modeling Radioactive Decay

Suppose the half-life of a certain radioactive substance is 20 days and that there are 5 grams present initially. When will there be only 1 gram of the substance remaining?

SOLUTION

Model The number of grams remaining after 20 days is

$$5\left(\frac{1}{2}\right) = \frac{5}{2}.$$

The number of grams remaining after 40 days is

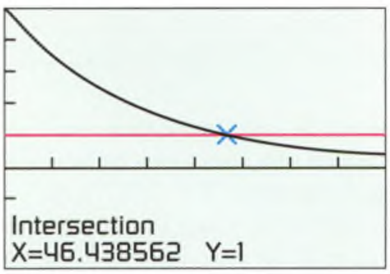
$$5\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 5\left(\frac{1}{2}\right)^2 = \frac{5}{4}.$$

The function $y = 5(1/2)^{t/20}$ models the mass in grams of the radioactive substance after t days.

Solve Graphically Figure 1.25 shows that the graphs of $y_1 = 5(1/2)^{t/20}$ and $y_2 = 1$ (for 1 gram) intersect when t is approximately 46.44.

Interpret There will be 1 gram of the radioactive substance left after approximately 46.44 days, or about 46 days 10.5 hours. *Now try Exercise 23.*

$y = 5\left(\frac{1}{2}\right)^{t/20}$, $y = 1$



[0, 80] by [–3, 5]

Figure 1.25 (Example 4)

Table 1.8 U.S. Population

Year	Population (millions)
1880	50.2
1890	63.0
1900	76.2
1910	92.2
1920	106.0
1930	123.2
1940	132.1
1950	151.3
1960	179.3
1970	203.3
1980	226.5
1990	248.7

Source: The Statistical Abstract of the United States, 2004–2005.

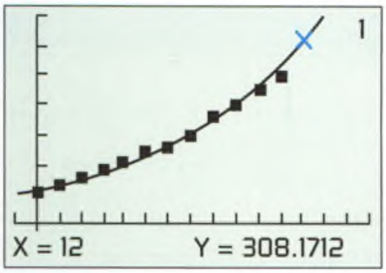
Compound interest investments, population growth, and radioactive decay are all examples of *exponential growth and decay.*

DEFINITIONS Exponential Growth, Exponential Decay

The function $y = k \cdot a^x$, $k > 0$ is a model for **exponential growth** if $a > 1$, and a model for **exponential decay** if $0 < a < 1$.

Applications

Most graphers have the exponential growth and decay model $y = k \cdot a^x$ built in as an exponential regression equation. We use this feature in Example 5 to analyze the U.S. population from the data in Table 1.8.



[–1, 15] by [–50, 350]

Figure 1.26 (Example 5)

EXAMPLE 5 Predicting the U.S. Population

Use the population data in Table 1.8 to estimate the population for the year 2000. Compare the result with the actual 2000 population of approximately 281.4 million.

SOLUTION

Model Let $x = 0$ represent 1880, $x = 1$ represent 1890, and so on. We enter the data into the grapher and find the exponential regression equation to be

$$f(x) = (56.4696)(1.1519)^x.$$

Figure 1.26 shows the graph of f superimposed on the scatter plot of the data.

Solve Graphically The year 2000 is represented by $x = 12$. Reading from the curve, we find

$$f(12) \approx 308.2.$$

The exponential model estimates the 2000 population to be 308.2 million, an overestimate of approximately 26.8 million, or about 9.5%.

Now try Exercise 39(a, b).

EXAMPLE 6 Interpreting Exponential Regression

What *annual* rate of growth can we infer from the exponential regression equation in Example 5?

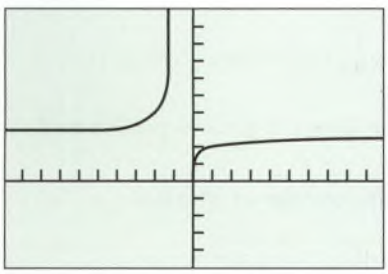
SOLUTION

Let r be the annual rate of growth of the U.S. population, expressed as a decimal. Because the time increments we used were 10-year intervals, we have

$$(1 + r)^{10} \approx 1.1519$$
$$r \approx \sqrt[10]{1.1519} - 1$$
$$r \approx 0.014$$

The annual rate of growth is about 1.4%. *Now try Exercise 39(c).*

$y = (1 + 1/x)^x$



[–10, 10] by [–5, 10]

X	Y1	
1000	2.7169	
2000	2.7176	
3000	2.7178	
4000	2.7179	
5000	2.718	
6000	2.7181	
7000	2.7181	

Y1 = (1+1/X)^X

Figure 1.27 A graph and table of values for $f(x) = (1 + 1/x)^x$ both suggest that as $x \to \infty$, $f(x) \to e \approx 2.718$.

The Number *e*

Many natural, physical, and economic phenomena are best modeled by an exponential function whose base is the famous number e, which is 2.718281828 to nine decimal places. We can define e to be the number that the function $f(x) = (1 + 1/x)^x$ approaches as x approaches infinity. The graph and table in Figure 1.27 strongly suggest that such a number exists.

The exponential functions $y = e^x$ and $y = e^{-x}$ are frequently used as models of exponential growth or decay. For example, interest **compounded continuously** uses the model $y = P \cdot e^{rt}$, where P is the initial investment, r is the interest rate as a decimal, and t is time in years.

Quick Review 1.3 *(For help, go to Section 1.3.)*

In Exercises 1–3, evaluate the expression. Round your answers to 3 decimal places.

1. $5^{2/3}$ **2.** $3^{\sqrt{2}}$

3. $3^{-1.5}$

In Exercises 4–6, solve the equation. Round your answers to 4 decimal places.

4. $x^3 = 17$ **5.** $x^5 = 24$

6. $x^{10} = 1.4567$

In Exercises 7 and 8, find the value of investing P dollars for n years with the interest rate r compounded annually.

7. $P = \$500,\quad r = 4.75\%,\quad n = 5$ years

8. $P = \$1000,\quad r = 6.3\%,\quad n = 3$ years

In Exercises 9 and 10, simplify the exponential expression.

9. $\dfrac{(x^{-3}y^2)^2}{(x^4y^3)^3}$ **10.** $\left(\dfrac{a^3b^{-2}}{c^4}\right)^2\left(\dfrac{a^4c^{-2}}{b^3}\right)^{-1}$

Section 1.3 Exercises

In Exercises 1–4, graph the function. State its domain and range.

1. $y = -2^x + 3$ **2.** $y = e^x + 3$

3. $y = 3 \cdot e^{-x} - 2$ **4.** $y = -2^{-x} - 1$

In Exercises 5–8, rewrite the exponential expression to have the indicated base.

5. 9^{2x}, base 3 **6.** 16^{3x}, base 2

7. $(1/8)^{2x}$, base 2 **8.** $(1/27)^x$, base 3

In Exercises 9–12, use a graph to find the zeros of the function.

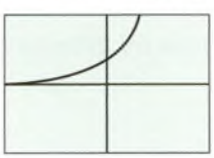
9. $f(x) = 2^x - 5$ **10.** $f(x) = e^x - 4$

11. $f(x) = 3^x - 0.5$ **12.** $f(x) = 3 - 2^x$

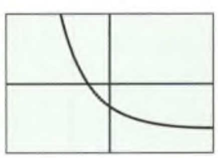
In Exercises 13–18, match the function with its graph. Try to do it without using your grapher.

13. $y = 2^x$ **14.** $y = 3^{-x}$ **15.** $y = -3^{-x}$

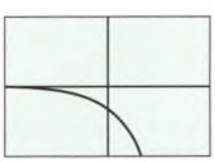
16. $y = -0.5^{-x}$ **17.** $y = 2^{-x} - 2$ **18.** $y = 1.5^x - 2$



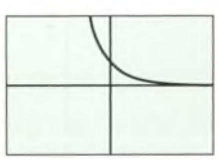
(a)



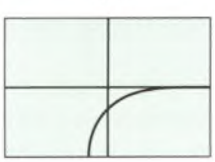
(b)



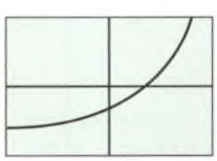
(c)



(d)



(e)



(f)

19. ***Population of Nevada*** Table 1.9 gives the population of Nevada for several years.

Table 1.9 Population of Nevada

Year	Population (thousands)
1998	1,853
1999	1,935
2000	1,998
2001	2,095
2002	2,167
2003	2,241

Source: Statistical Abstract of the United States, 2004–2005.

(a) Compute the ratios of the population in one year by the population in the previous year.

(b) Based on part (a), create an exponential model for the population of Nevada.

(c) Use your model in part (b) to predict the population of Nevada in 2010.

20. ***Population of Virginia*** Table 1.10 gives the population of Virginia for several years.

Table 1.10 Population of Virginia

Year	Population (thousands)
1998	6,901
1999	7,000
2000	7,078
2001	7,193
2002	7,288
2003	7,386

Source: Statistical Abstract of the United States, 2004–2005.

(a) Compute the ratios of the population in one year by the population in the previous year.

(b) Based on part (a), create an exponential model for the population of Virginia.

(c) Use your model in part (b) to predict the population of Virginia in 2008.

In Exercises 21–32, use an exponential model to solve the problem.

21. ***Population Growth*** The population of Knoxville is 500,000 and is increasing at the rate of 3.75% each year. Approximately when will the population reach 1 million?

22. ***Population Growth*** The population of Silver Run in the year 1890 was 6250. Assume the population increased at a rate of 2.75% per year.

(a) Estimate the population in 1915 and 1940.

(b) Approximately when did the population reach 50,000?

23. ***Radioactive Decay*** The half-life of phosphorus-32 is about 14 days. There are 6.6 grams present initially.

(a) Express the amount of phosphorus-32 remaining as a function of time t.

(b) When will there be 1 gram remaining?

24. ***Finding Time*** If John invests $2300 in a savings account with a 6% interest rate compounded annually, how long will it take until John's account has a balance of $4150?

25. ***Doubling Your Money*** Determine how much time is required for an investment to double in value if interest is earned at the rate of 6.25% compounded annually.

26. ***Doubling Your Money*** Determine how much time is required for an investment to double in value if interest is earned at the rate of 6.25% compounded monthly.

27. ***Doubling Your Money*** Determine how much time is required for an investment to double in value if interest is earned at the rate of 6.25% compounded continuously.

28. ***Tripling Your Money*** Determine how much time is required for an investment to triple in value if interest is earned at the rate of 5.75% compounded annually.

29. ***Tripling Your Money*** Determine how much time is required for an investment to triple in value if interest is earned at the rate of 5.75% compounded daily.

30. ***Tripling Your Money*** Determine how much time is required for an investment to triple in value if interest is earned at the rate of 5.75% compounded continuously.

31. ***Cholera Bacteria*** Suppose that a colony of bacteria starts with 1 bacterium and doubles in number every half hour. How many bacteria will the colony contain at the end of 24 h?

32. ***Eliminating a Disease*** Suppose that in any given year, the number of cases of a disease is reduced by 20%. If there are 10,000 cases today, how many years will it take

(a) to reduce the number of cases to 1000?

(b) to eliminate the disease; that is, to reduce the number of cases to less than 1?

Group Activity In Exercises 33–36, copy and complete the table for the function.

33. $y = 2x - 3$

x	y	Change (Δy)
1	?	
2	?	?
3	?	?
4	?	?

34. $y = -3x + 4$

x	y	Change (Δy)
1	?	
2	?	?
3	?	?
4	?	?

35. $y = x^2$

x	y	Change (Δy)
1	?	
2	?	?
3	?	?
4	?	?

36. $y = 3e^x$

x	y	Ratio (y_i/y_{i-1})
1	?	
2	?	?
3	?	?
4	?	?

37. **Writing to Learn** Explain how the change Δy is related to the slopes of the lines in Exercises 33 and 34. If the changes in x are constant for a linear function, what would you conclude about the corresponding changes in y?

38. ***Bacteria Growth*** The number of bacteria in a petri dish culture after t hours is

$$B = 100e^{0.693t}.$$

(a) What was the initial number of bacteria present?

(b) How many bacteria are present after 6 hours?

(c) Approximately when will the number of bacteria be 200? Estimate the doubling time of the bacteria.

39. ***Population of Texas*** Table 1.11 gives the population of Texas for several years.

Table 1.11 Population of Texas

Year	Population (thousands)
1980	14,229
1990	16,986
1995	18,959
1998	20,158
1999	20,558
2000	20,852

Source: Statistical Abstract of the United States, 2004-2005.

(a) Let $x = 0$ represent 1980, $x = 1$ represent 1981, and so forth. Find an exponential regression for the data, and superimpose its graph on a scatter plot of the data.

(b) Use the exponential regression equation to estimate the population of Texas in 2003. How close is the estimate to the actual population of 22,119,000 in 2003?

(c) Use the exponential regression equation to estimate the annual rate of growth of the population of Texas.

40. ***Population of California*** Table 1.12 gives the population of California for several years.

Table 1.12 Population of California

Year	Population (thousands)
1980	23,668
1990	29,811
1995	31,697
1998	32,988
1999	33,499
2000	33,872

Source: Statistical Abstract of the United States, 2004-2005.

(a) Let $x = 0$ represent 1980, $x = 1$ represent 1981, and so forth. Find an exponential regression for the data, and superimpose its graph on a scatter plot of the data.

(b) Use the exponential regression equation to estimate the population of California in 2003. How close is the estimate to the actual population of 35,484,000 in 2003?

(c) Use the exponential regression equation to estimate the annual rate of growth of the population of California.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

41. True or False The number 3^{-2} is negative. Justify your answer.

42. True or False If $4^3 = 2^a$, then $a = 6$. Justify your answer.

43. Multiple Choice John invests $200 at 4.5% compounded annually. About how long will it take for John's investment to double in value?

(A) 6 yrs **(B)** 9 yrs **(C)** 12 yrs **(D)** 16 yrs **(E)** 20 yrs

44. Multiple Choice Which of the following gives the domain of $y = 2e^{-x} - 3$?

(A) $(-\infty, \infty)$ **(B)** $[-3, \infty)$ **(C)** $[-1, \infty)$ **(D)** $(-\infty, 3]$
(E) $x \neq 0$

45. Multiple Choice Which of the following gives the range of $y = 4 - 2^{-x}$?

(A) $(-\infty, \infty)$ **(B)** $(-\infty, 4)$ **(C)** $[-4, \infty)$
(D) $(-\infty, 4]$ **(E)** all reals

46. Multiple Choice Which of the following gives the best approximation for the zero of $f(x) = 4 - e^x$?

(A) $x = -1.386$ **(B)** $x = 0.386$ **(C)** $x = 1.386$
(D) $x = 3$ **(E)** there are no zeros

Exploration

47. Let $y_1 = x^2$ and $y_2 = 2^x$.

(a) Graph y_1 and y_2 in $[-5, 5]$ by $[-2, 10]$. How many times do you think the two graphs cross?

(b) Compare the corresponding changes in y_1 and y_2 as x changes from 1 to 2, 2 to 3, and so on. How large must x be for the changes in y_2 to overtake the changes in y_1?

(c) Solve for x: $x^2 = 2^x$. **(d)** Solve for x: $x^2 < 2^x$.

Extending the Ideas

In Exercises 48 and 49, assume that the graph of the exponential function $f(x) = k \cdot a^x$ passes through the two points. Find the values of a and k.

48. $(1, 4.5), (-1, 0.5)$ **49.** $(1, 1.5), (-1, 6)$

Quick Quiz for AP* Preparation: Sections 1.1–1.3

You may use graphing calculator to solve the following problems.

1. **Multiple Choice** Which of the following gives an equation for the line through $(3, -1)$ and parallel to the line $y = -2x + 1$?

   (A) $= \frac{1}{2}x + \frac{7}{2}$ **(B)** $y = \frac{1}{2}x - \frac{5}{2}$ **(C)** $y = -2x + 5$

   (D) $y = -2x - 7$ **(E)** $y = -2x + 1$

2. **Multiple Choice** If $f(x) = x^2 + 1$ and $g(x) = 2x - 1$, which of the following gives $f \circ g(2)$?

   (A) 2 **(B)** 5 **(C)** 9 **(D)** 10 **(E)** 15

3. **Multiple Choice** The half-life of a certain radioactive substance is 8 hrs. There are 5 grams present initially. Which of the following gives the best approximation when there will be 1 gram remaining?

   (A) 2 **(B)**10 **(C)** 15 **(D)** 16 **(E)** 19

4. **Free Response** Let $f(x) = e^{-x} - 2$.

   (a) Find the domain of f. **(b)** Find the range of f.

   (c) Find the zeros of f.

1.4 Parametric Equations

What you'll learn about

- Relations
- Circles
- Ellipses
- Lines and Other Curves

. . . and why

Parametric equations can be used to obtain graphs of relations and functions.

Relations

A **relation** is a set of ordered pairs (x, y) of real numbers. The **graph of a relation** is the set of points in the plane that correspond to the ordered pairs of the relation. If x and y are *functions* of a third variable t, called a *parameter*, then we can use the *parametric mode* of a grapher to obtain a graph of the relation.

EXAMPLE 1 Graphing Half a Parabola

Describe the graph of the relation determined by

$$x = \sqrt{t}, \quad y = t, \quad t \geq 0.$$

Indicate the direction in which the curve is traced. Find a Cartesian equation for a curve that contains the parametrized curve.

SOLUTION

Set $x_1 = \sqrt{t}$, $y_1 = t$, and use the parametric mode of the grapher to draw the graph in Figure 1.28. The graph appears to be the right half of the parabola $y = x^2$. Notice that there is no information about t on the graph itself. The curve appears to be traced to the upper right with starting point (0, 0).

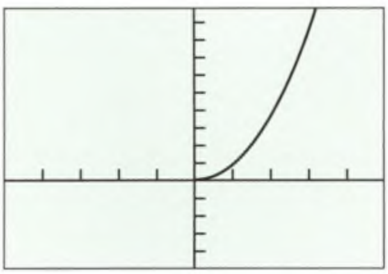
Confirm Algebraically Both x and y will be greater than or equal to zero because $t \geq 0$. Eliminating t we find that for every value of t,

$$y = t = (\sqrt{t})^2 = x^2.$$

Thus, the relation is the function $y = x^2$, $x \geq 0$.

Now try Exercise 5.

$x = \sqrt{t}$, $y = t$



[–5, 5] by [–5, 10]

Figure 1.28 You must choose a *smallest* and *largest* value for t in parametric mode. Here we used 0 and 10, respectively. (Example 1)

DEFINITIONS Parametric Curve, Parametric Equations

If x and y are given as functions

$$x = f(t), \quad y = g(t)$$

over an interval of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve.** The equations are **parametric equations** for the curve.

The variable t is a **parameter** for the curve and its domain I is the **parameter interval.** If I is a closed interval, $a \leq t \leq b$, the point $(f(a), g(a))$ is the **initial point of the curve** and the point $(f(b), g(b))$ is the **terminal point of the curve.** When we give parametric equations and a parameter interval for a curve, we say that we have **parametrized** the curve. The equations and interval constitute a **parametrization of the curve.**

In Example 1, the parameter interval is $[0, \infty)$, so (0, 0) is the initial point and there is no terminal point.

A grapher can draw a parametrized curve only over a closed interval, so the portion it draws has endpoints even when the curve being graphed does not. Keep this in mind when you graph.

Circles

In applications, t often denotes time, an angle, or the distance a particle has traveled along its path from its starting point. In fact, parametric graphing can be used to simulate the motion of the particle.

EXPLORATION 1 **Parametrizing Circles**

Let $x = a\cos t$ and $y = a\sin t$.

1. Let $a = 1, 2,$ or 3 and graph the parametric equations in a *square viewing window* using the parameter interval $[0, 2\pi]$. How does changing a affect this graph?
2. Let $a = 2$ and graph the parametric equations using the following parameter intervals: $[0, \pi/2]$, $[0, \pi]$, $[0, 3\pi/2]$, $[2\pi, 4\pi]$, and $[0, 4\pi]$. Describe the role of the length of the parameter interval.
3. Let $a = 3$ and graph the parametric equations using the intervals $[\pi/2, 3\pi/2]$, $[\pi, 2\pi]$, $[3\pi/2, 3\pi]$, and $[\pi, 5\pi]$. What are the initial point and terminal point in each case?
4. Graph $x = 2\cos(-t)$ and $y = 2\sin(-t)$ using the parameter intervals $[0, 2\pi]$, $[\pi, 3\pi]$, and $[\pi/2, 3\pi/2]$. In each case, describe how the graph is traced.

For $x = a\cos t$ and $y = a\sin t$, we have

$$x^2 + y^2 = a^2\cos^2 t + a^2\sin^2 t = a^2(\cos^2 t + \sin^2 t) = a^2(1) = a^2,$$

using the identity $\cos^2 t + \sin^2 t = 1$. Thus, the curves in Exploration 1 were either circles or portions of circles, each with center at the origin.

EXAMPLE 2 Graphing a Circle

Describe the graph of the relation determined by

$$x = 2\cos t, \quad y = 2\sin t, \quad 0 \le t \le 2\pi.$$

Find the initial and terminal points, if any, and indicate the direction in which the curve is traced. Find a Cartesian equation for a curve that contains the parametrized curve.

SOLUTION

Figure 1.29 shows that the graph appears to be a circle with radius 2. By watching the graph develop we can see that the curve is traced exactly once counterclockwise. The initial point at $t = 0$ is $(2, 0)$, and the terminal point at $t = 2\pi$ is also $(2, 0)$.

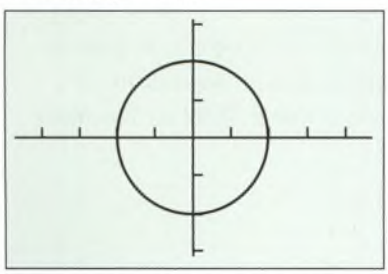
Next we eliminate the variable t.

$$\begin{aligned} x^2 + y^2 &= 4\cos^2 t + 4\sin^2 t \\ &= 4(\cos^2 t + \sin^2 t) \\ &= 4 \qquad \text{Because } \cos^2 t + \sin^2 t = 1 \end{aligned}$$

The parametrized curve is a circle centered at the origin of radius 2.

Now try Exercise 9.

$x = 2\cos t,\ y = 2\sin t$



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

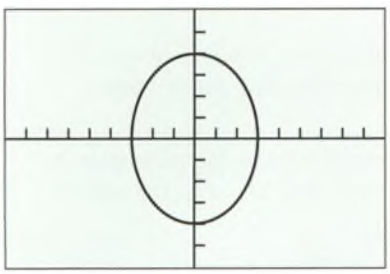
Figure 1.29 A graph of the parametric curve $x = 2\cos t$, $y = 2\sin t$, with Tmin $= 0$, Tmax $= 2\pi$, and Tstep $= \pi/24 \approx 0.131$. (Example 2)

Ellipses

Parametrizations of ellipses are similar to parametrizations of circles. Recall that the standard form of an ellipse centered at (0, 0) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

$x = 3\cos t,\ y = 4\sin t$



[–9, 9] by [–6, 6]

Figure 1.30 A graph of the parametric equations $x = 3\cos t$, $y = 4\sin t$ for $0 \le t \le 2\pi$. (Example 3)

EXAMPLE 3 Graphing an Ellipse

Graph the parametric curve $x = 3\cos t$, $y = 4\sin t$, $0 \le t \le 2\pi$.

Find a Cartesian equation for a curve that contains the parametric curve. What portion of the graph of the Cartesian equation is traced by the parametric curve? Indicate the direction in which the curve is traced and the initial and terminal points, if any.

SOLUTION

Figure 1.30 suggests that the curve is an ellipse. The Cartesian equation is

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = \cos^2 t + \sin^2 t = 1,$$

so the parametrized curve lies along an ellipse with major axis endpoints $(0, \pm 4)$ and minor axis endpoints $(\pm 3, 0)$. As t increases from 0 to 2π, the point $(x, y) = (3\cos t, 4\sin t)$ starts at (3, 0) and traces the entire ellipse once counterclockwise. Thus, (3, 0) is both the initial point and the terminal point. *Now try Exercise 13.*

EXPLORATION 2 Parametrizing Ellipses

Let $x = a\cos t$ and $y = b\sin t$.

1. Let $a = 2$ and $b = 3$. Then graph using the parameter interval $[0, 2\pi]$. Repeat, changing b to 4, 5, and 6.
2. Let $a = 3$ and $b = 4$. Then graph using the parameter interval $[0, 2\pi]$. Repeat, changing a to 5, 6, and 7.
3. Based on parts 1 and 2, how do you identify the axis that contains the major axis of the ellipse? the minor axis?
4. Let $a = 4$ and $b = 3$. Then graph using the parameter intervals $[0, \pi/2]$, $[0, \pi]$, $[0, 3\pi/2]$, and $[0, 4\pi]$. Describe the role of the length of the parameter interval.
5. Graph $x = 5\cos(-t)$ and $y = 2\sin(-t)$ using the parameter intervals $(0, 2\pi]$, $[\pi, 3\pi]$, and $[\pi/2, 3\pi/2]$. Describe how the graph is traced. What are the initial point and terminal point in each case?

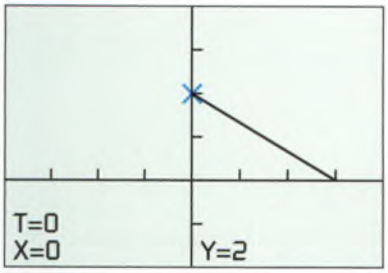
For $x = a\cos t$ and $y = b\sin t$, we have $(x/a)^2 + (y/b)^2 = \cos^2 t + \sin^2 t = 1$. Thus, the curves in Exploration 2 were either ellipses or portions of ellipses, each with center at the origin.

In the exercises you will see how to graph hyperbolas parametrically.

Lines and Other Curves

Lines, line segments, and many other curves can be defined parametrically.

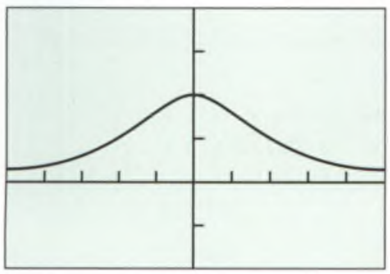
$x = 3t,\ y = 2 - 2t$



[−4, 4] by [−2, 4]

Figure 1.31 The graph of the line segment $x = 3t$, $y = 2 - 2t$, $0 \le t \le 1$, with trace on the initial point (0, 2). (Example 4)

$x = 2\cot t,\ y = 2\sin^2 t$



[−5, 5] by [−2, 4]

Figure 1.32 The witch of Agnesi (Exploration 3)

Maria Agnesi (1718–1799)

The first text to include differential and integral calculus along with analytic geometry, infinite series, and differential equations was written in the 1740s by the Italian mathematician Maria Gaetana Agnesi. Agnesi, a gifted scholar and linguist whose Latin essay defending higher education for women was published when she was only nine years old, was a well-published scientist by age 20, and an honorary faculty member of the University of Bologna by age 30.

Today, Agnesi is remembered chiefly for a bell-shaped curve called *the witch of Agnesi*. This name, found only in English texts, is the result of a mistranslation. Agnesi's own name for the curve was *versiera* or "turning curve." John Colson, a noted Cambridge mathematician, probably confused versiera with *avversiera*, which means "wife of the devil" and translated it into "witch."

EXAMPLE 4 Graphing a Line Segment

Draw and identify the graph of the parametric curve determined by

$$x = 3t, \quad y = 2 - 2t, \quad 0 \le t \le 1.$$

SOLUTION

The graph (Figure 1.31) appears to be a line segment with endpoints (0, 2) and (3, 0).

Confirm Algebraically When $t = 0$, the equations give $x = 0$ and $y = 2$. When $t = 1$, they give $x = 3$ and $y = 0$. When we substitute $t = x/3$ into the y equation, we obtain

$$y = 2 - 2\left(\frac{x}{3}\right) = -\frac{2}{3}x + 2.$$

Thus, the parametric curve traces the segment of the line $y = -(2/3)x + 2$ from the point (0, 2) to (3, 0).

Now try Exercise 17.

If we change the parameter interval [0, 1] in Example 4 to $(-\infty, \infty)$, the parametrization will trace the entire line $y = -(2/3)x + 2$.

The bell-shaped curve in Exploration 3 is the famous witch of Agnesi. You will find more information about this curve in Exercise 47.

EXPLORATION 3 Graphing the Witch of Agnesi

The witch of Agnesi is the curve

$$x = 2\cot t, \quad y = 2\sin^2 t, \quad 0 < t < \pi.$$

1. Draw the curve using the window in Figure 1.32. What did you choose as a closed parameter interval for your grapher? In what direction is the curve traced? How far to the left and right of the origin do you think the curve extends?
2. Graph the same parametric equations using the parameter intervals $(-\pi/2, \pi/2)$, $(0, \pi/2)$, and $(\pi/2, \pi)$. In each case, describe the curve you see and the direction in which it is traced by your grapher.
3. What happens if you replace $x = 2\cot t$ by $x = -2\cot t$ in the original parametrization? What happens if you use $x = 2\cot(\pi - t)$?

EXAMPLE 5 Parametrizing a Line Segment

Find a parametrization for the line segment with endpoints $(-2, 1)$ and $(3, 5)$.

SOLUTION

Using $(-2, 1)$ we create the parametric equations

$$x = -2 + at, \qquad y = 1 + bt.$$

These represent a line, as we can see by solving each equation for t and equating to obtain

$$\frac{x + 2}{a} = \frac{y - 1}{b}.$$

continued

This line goes through the point $(-2, 1)$ when $t = 0$. We determine a and b so that the line goes through $(3, 5)$ when $t = 1$.

$$3 = -2 + a \quad \Rightarrow \quad a = 5 \qquad x = 3 \text{ when } t = 1.$$

$$5 = 1 + b \quad \Rightarrow \quad b = 4 \qquad y = 5 \text{ when } t = 1.$$

Therefore,

$$x = -2 + 5t, \quad y = 1 + 4t, \quad 0 \le t \le 1$$

is a parametrization of the line segment with initial point $(-2, 1)$ and terminal point $(3, 5)$.

Now try Exercise 23.

Quick Review 1.4 *(For help, go to Section 1.1 and Appendix A1.)*

In Exercises 1–3, write an equation for the line.

1. the line through the points $(1, 8)$ and $(4, 3)$

2. the horizontal line through the point $(3, -4)$

3. the vertical line through the point $(2, -3)$

In Exercises 4–6, find the x- and y-intercepts of the graph of the relation.

4. $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$ **5.** $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$

6. $2y^2 = x + 1$

In Exercises 7 and 8, determine whether the given points lie on the graph of the relation.

7. $2x^2y + y^2 = 3$

(a) $(1, 1)$ **(b)** $(-1, -1)$ **(c)** $(1/2, -2)$

8. $9x^2 - 18x + 4y^2 = 27$

(a) $(1, 3)$ **(b)** $(1, -3)$ **(c)** $(-1, 3)$

9. Solve for t.

(a) $2x + 3t = -5$ **(b)** $3y - 2t = -1$

10. For what values of a is each equation true?

(a) $\sqrt{a^2} = a$ **(b)** $\sqrt{a^2} = \pm a$ **(c)** $\sqrt{4a^2} = 2|a|$

Section 1.4 Exercises

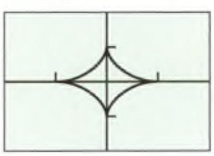
In Exercises 1–4, match the parametric equations with their graph. State the approximate dimensions of the viewing window. Give a parameter interval that traces the curve exactly once.

1. $x = 3\sin(2t), \quad y = 1.5\cos t$

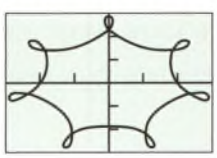
2. $x = \sin^3 t, \quad y = \cos^3 t$

3. $x = 7\sin t - \sin(7t), \quad y = 7\cos t - \cos(7t)$

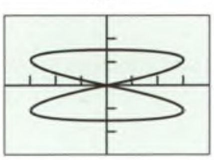
4. $x = 12\sin t - 3\sin(6t), \quad y = 12\cos t + 3\cos(6t)$



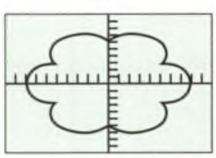
(a)



(b)



(c)



(d)

In Exercises 5–22, a parametrization is given for a curve.

(a) Graph the curve. What are the initial and terminal points, if any? Indicate the direction in which the curve is traced.

(b) Find a Cartesian equation for a curve that contains the parametrized curve. What portion of the graph of the Cartesian equation is traced by the parametrized curve?

5. $x = 3t, \quad y = 9t^2, \quad -\infty < t < \infty$

6. $x = -\sqrt{t}, \quad y = t, \quad t \ge 0$

7. $x = t, \quad y = \sqrt{t}, \quad t \ge 0$

8. $x = (\sec^2 t) - 1, \quad y = \tan t, \quad -\pi/2 < t < \pi/2$

9. $x = \cos t, \quad y = \sin t, \quad 0 \le t \le \pi$

10. $x = \sin(2\pi t), \quad y = \cos(2\pi t), \quad 0 \le t \le 1$

11. $x = \cos(\pi - t), \quad y = \sin(\pi - t), \quad 0 \le t \le \pi$

12. $x = 4\cos t, \quad y = 2\sin t, \quad 0 \le t \le 2\pi$

13. $x = 4\sin t, \quad y = 2\cos t, \quad 0 \le t \le \pi$

14. $x = 4\sin t, \quad y = 5\cos t, \quad 0 \le t \le 2\pi$

15. $x = 2t - 5, \quad y = 4t - 7, \quad -\infty < t < \infty$

16. $x = 1 - t,\quad y = 1 + t,\quad -\infty < t < \infty$

17. $x = t,\quad y = 1 - t,\quad 0 \le t \le 1$

18. $x = 3 - 3t,\quad y = 2t,\quad 0 \le t \le 1$

19. $x = 4 - \sqrt{t},\quad y = \sqrt{t},\quad 0 \le t$

20. $x = t^2,\quad y = \sqrt{4 - t^2},\quad 0 \le t \le 2$

21. $x = \sin t,\quad y = \cos 2t,\quad -\infty < t < \infty$

22. $x = t^2 - 3,\quad y = t,\quad t \le 0$

In Exercises 23–28, find a parametrization for the curve.

23. the line segment with endpoints $(-1, -3)$ and $(4, 1)$

24. the line segment with endpoints $(-1, 3)$ and $(3, -2)$

25. the lower half of the parabola $x - 1 = y^2$

26. the left half of the parabola $y = x^2 + 2x$

27. the ray (half line) with initial point $(2, 3)$ that passes through the point $(-1, -1)$

28. the ray (half line) with initial point $(-1, 2)$ that passes through the point $(0, 0)$

Group Activity In Exercises 29–32, refer to the graph of

$$x = 3 - |t|,\quad y = t - 1,\quad -5 \le t \le 5,$$

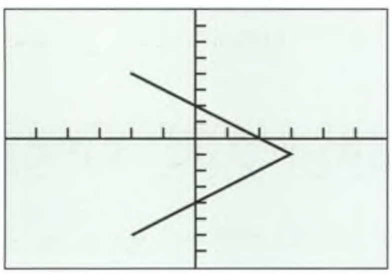
shown in the figure. Find the values of t that produce the graph in the given quadrant.

29. Quadrant I

30. Quadrant II

31. Quadrant III

32. Quadrant IV



[-6, 6] by [-8, 8]

In Exercises 33 and 34, find a parametrization for the part of the graph that lies in Quadrant I.

33. $y = x^2 + 2x + 2$

34. $y = \sqrt{x + 3}$

35. ***Circles*** Find parametrizations to model the motion of a particle that starts at $(a, 0)$ and traces the circle $x^2 + y^2 = a^2$, $a > 0$, as indicated.

(a) once clockwise
(b) once counterclockwise
(c) twice clockwise
(d) twice counterclockwise

36. ***Ellipses*** Find parametrizations to model the motion of a particle that starts at $(-a, 0)$ and traces the ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1,\quad a > 0, b > 0,$$

as indicated.

(a) once clockwise
(b) once counterclockwise
(c) twice clockwise
(d) twice counterclockwise

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

37. True or False The graph of the parametric curve $x = 3 \cos t$, $y = 4 \sin t$ is a circle. Justify your answer.

38. True or False The parametric curve $x = 2 \cos(-t)$, $y = 2 \sin(-t)$, $0 \le t \le 2\pi$ is traced clockwise. Justify your answer.

In Exercises 39 and 40, use the parametric curve $x = 5t$, $y = 3 - 3t$, $0 \le t \le 1$.

39. Multiple Choice Which of the following describes its graph?

(A) circle **(B)** parabola **(C)** ellipse
(D) line segment **(E)** line

40. Multiple Choice Which of the following is the initial point of the curve?

(A) $(-5, 6)$ **(B)** $(0, -3)$ **(C)** $(0, 3)$ **(D)** $(5, 0)$
(E) $(10, -3)$

41. Multiple Choice Which of the following describes the graph of the parametric curve $x = -3 \sin t$, $y = -3 \cos t$?

(A) circle **(B)** parabola **(C)** ellipse
(D) hyperbola **(E)** line

42. Multiple Choice Which of the following describes the graph of the parametric curve $x = 3t$, $y = 2t$, $t \ge 1$?

(A) circle **(B)** parabola **(C)** line segment
(D) line **(E)** ray

Explorations

43. ***Hyperbolas*** Let $x = a \sec t$ and $y = b \tan t$.

(a) Writing to Learn Let $a = 1, 2,$ or 3, $b = 1, 2,$ or 3, and graph using the parameter interval $(-\pi/2, \pi/2)$. Explain what you see, and describe the role of a and b in these parametric equations. (Caution: If you get what appear to be asymptotes, try using the approximation $[-1.57, 1.57]$ for the parameter interval.)

(b) Let $a = 2$, $b = 3$, and graph in the parameter interval $(\pi/2, 3\pi/2)$. Explain what you see.

(c) Writing to Learn Let $a = 2$, $b = 3$, and graph using the parameter interval $(-\pi/2, 3\pi/2)$. Explain why you must be careful about graphing in this interval or any interval that contains $\pm\pi/2$.

(d) Use algebra to explain why

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1.$$

(e) Let $x = a \tan t$ and $y = b \sec t$. Repeat (a), (b), and (d) using an appropriate version of (d).

44. ***Transformations*** Let $x = (2 \cos t) + h$ and $y = (2 \sin t) + k$.

(a) Writing to Learn Let $k = 0$ and $h = -2, -1, 1,$ and 2, in turn. Graph using the parameter interval $[0, 2\pi]$. Describe the role of h.

(b) Writing to Learn Let $h = 0$ and $k = -2, -1, 1$, and 2, in turn. Graph using the parameter interval $[0, 2\pi]$. Describe the role of k.

(c) Find a parametrization for the circle with radius 5 and center at $(2, -3)$.

(d) Find a parametrization for the ellipse centered at $(-3, 4)$ with semimajor axis of length 5 parallel to the x-axis and semiminor axis of length 2 parallel to the y-axis.

In Exercises 45 and 46, a parametrization is given for a curve.

(a) Graph the curve. What are the initial and terminal points, if any? Indicate the direction in which the curve is traced.

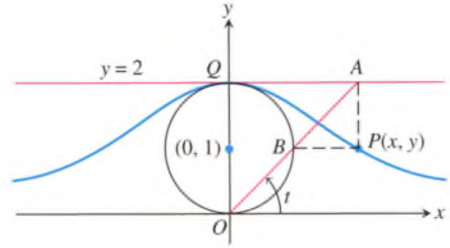
(b) Find a Cartesian equation for a curve that contains the parametrized curve. What portion of the graph of the Cartesian equation is traced by the parametrized curve?

45. $x = -\sec t, \quad y = \tan t, \quad -\pi/2 < t < \pi/2$

46. $x = \tan t, \quad y = -2 \sec t, \quad -\pi/2 < t < \pi/2$

Extending the Ideas

47. *The Witch of Agnesi* The bell-shaped witch of Agnesi can be constructed as follows. Start with the circle of radius 1, centered at the point (0, 1) as shown in the figure.



Choose a point A on the line $y = 2$, and connect it to the origin with a line segment. Call the point where the segment crosses the circle B. Let P be the point where the vertical line through A crosses the horizontal line through B. The witch is the curve traced by P as A moves along the line $y = 2$.

Find a parametrization for the witch by expressing the coordinates of P in terms of t, the radian measure of the angle that segment OA makes with the positive x-axis. The following equalities (which you may assume) will help:

(i) $x = AQ$ **(ii)** $y = 2 - AB \sin t$ **(iii)** $AB \cdot AO = (AQ)^2$

48. *Parametrizing Lines and Segments*

(a) Show that $x = x_1 + (x_2 - x_1)t, \quad y = y_1 + (y_2 - y_1)t,$ $-\infty < t < \infty$ is a parametrization for the line through the points (x_1, y_1) and (x_2, y_2).

(b) Find a parametrization for the line segment with endpoints (x_1, y_1) and (x_2, y_2).

1.5 Functions and Logarithms

What you'll learn about

- One-to-One Functions
- Inverses
- Finding Inverses
- Logarithmic Functions
- Properties of Logarithms
- Applications

. . . and why

Logarithmic functions are used in many applications, including finding time in investment problems.

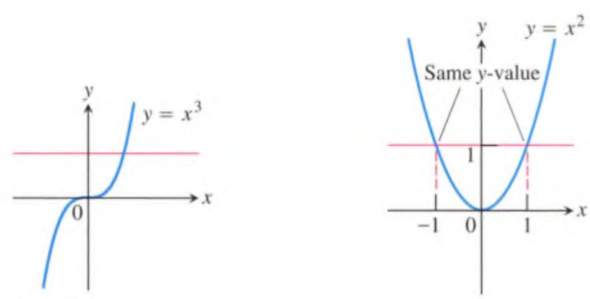
One-to-One Functions

As you know, a function is a rule that assigns a single value in its range to each point in its domain. Some functions assign the same output to more than one input. For example, $f(x) = x^2$ assigns the output 4 to both 2 and -2. Other functions never output a given value more than once. For example, the cubes of different numbers are always different.

If each output value of a function is associated with exactly one input value, the function is *one-to-one.*

> **DEFINITION One-to-One Function**
>
> A function $f(x)$ is **one-to-one** on a domain D if $f(a) \neq f(b)$ whenever $a \neq b$.

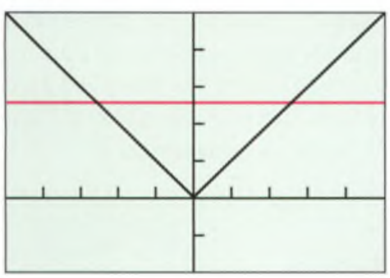
The graph of a one-to-one function $y = f(x)$ can intersect any horizontal line at most once (the *horizontal line test*). If it intersects such a line more than once it assumes the same y-value more than once, and is therefore not one-to-one (Figure 1.33).



One-to-one: Graph meets each horizontal line once. Not one-to-one: Graph meets some horizontal lines more than once.

Figure 1.33 Using the horizontal line test, we see that $y = x^3$ is one-to-one and $y = x^2$ is not.

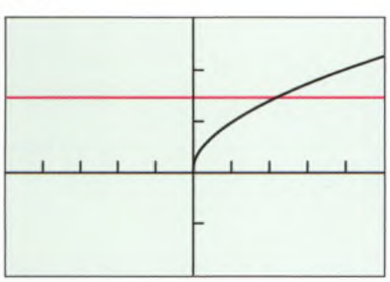
$y = |x|$



[–5, 5] by [–2, 5]

(a)

$y = \sqrt{x}$



[–5, 5] by [–2, 3]

(b)

Figure 1.34 (a) The graph of $f(x) = |x|$ and a horizontal line. (b) The graph of $g(x) = \sqrt{x}$ and a horizontal line. (Example 1)

EXAMPLE 1 Using the Horizontal Line Test

Determine whether the functions are one-to-one.

(a) $f(x) = |x|$ **(b)** $g(x) = \sqrt{x}$

SOLUTION

(a) As Figure 1.34a suggests, each horizontal line $y = c$, $c > 0$, intersects the graph of $f(x) = |x|$ twice. So f is not one-to-one.

(b) As Figure 1.34b suggests, each horizontal line intersects the graph of $g(x) = \sqrt{x}$ either once or not at all. The function g is one-to-one.

Now try Exercise 1.

Inverses

Since each output of a one-to-one function comes from just one input, a one-to-one function can be reversed to send outputs back to the inputs from which they came. The function defined by reversing a one-to-one function f is the **inverse of f.** The functions in Tables 1.13 and 1.14 are inverses of one another. The symbol for the inverse of f is f^{-1}, read "f inverse." The -1 in f^{-1} is not an exponent; $f^{-1}(x)$ does not mean $1/f(x)$.

Table 1.13 Rental Charge versus Time

Time x (hours)	Charge y (dollars)
1	5.00
2	7.50
3	10.00
4	12.50
5	15.00
6	17.50

Table 1.14 Time versus Rental Charge

Charge x (dollars)	Time y (hours)
5.00	1
7.50	2
10.00	3
12.50	4
15.00	5
17.50	6

As Tables 1.13 and 1.14 suggest, composing a function with its inverse in either order sends each output back to the input from which it came. In other words, the result of composing a function and its inverse in either order is the **identity function,** the function that assigns each number to itself. This gives a way to test whether two functions f and g are inverses of one another. Compute $f \circ g$ and $g \circ f$. If $(f \circ g)(x) = (g \circ f)(x) = x$, then f and g are inverses of one another; otherwise they are not. The functions $f(x) = x^3$ and $g(x) = x^{1/3}$ are inverses of one another because $(x^3)^{1/3} = x$ and $(x^{1/3})^3 = x$ for every number x.

EXPLORATION 1 Testing for Inverses Graphically

For each of the function pairs below,

(a) Graph f and g together in a square window.

(b) Graph $f \circ g$. **(c)** Graph $g \circ f$.

What can you conclude from the graphs?

1. $f(x) = x^3, \quad g(x) = x^{1/3}$

2. $f(x) = x, \quad g(x) = 1/x$

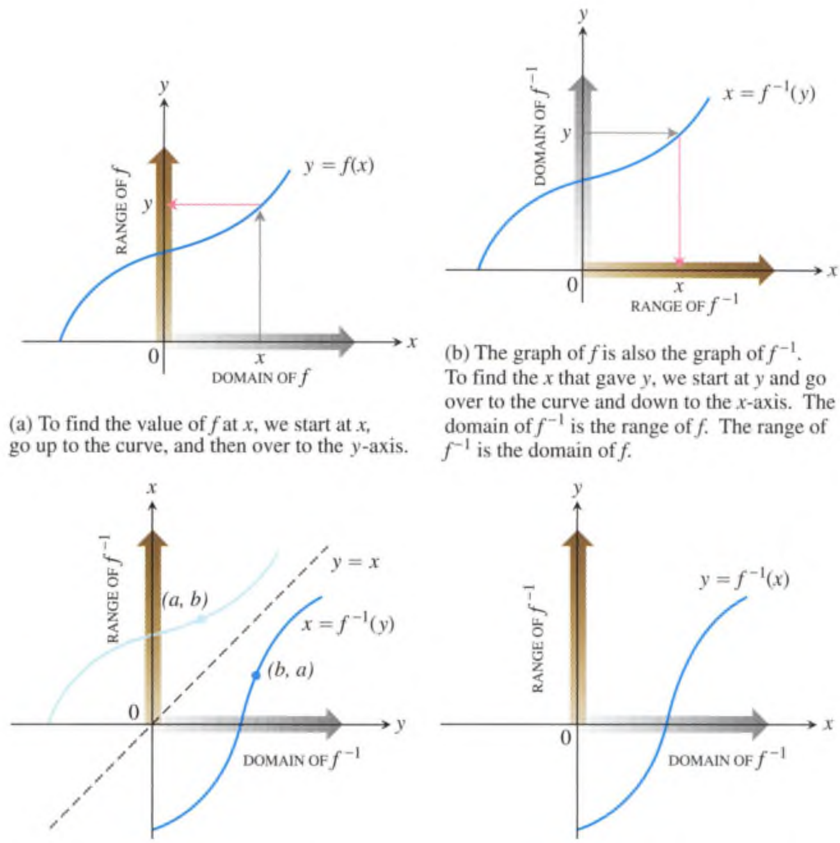
3. $f(x) = 3x, \quad g(x) = x/3$

4. $f(x) = e^x, \quad g(x) = \ln x$

Finding Inverses

How do we find the graph of the inverse of a function? Suppose, for example, that the function is the one pictured in Figure 1.35a. To read the graph, we start at the point x on the x-axis, go up to the graph, and then move over to the y-axis to read the value of y. If we start with y and want to find the x from which it came, we reverse the process (Figure 1.35b).

The graph of f is already the graph of f^{-1}, although the latter graph is not drawn in the usual way with the domain axis horizontal and the range axis vertical. For f^{-1}, the input-output pairs are reversed. To display the graph of f^{-1} in the usual way, we have to reverse the pairs by reflecting the graph across the 45° line $y = x$ (Figure 1.35c) and interchanging the letters x and y (Figure 1.35d). This puts the independent variable, now called x, on the horizontal axis and the dependent variable, now called y, on the vertical axis.



(a) To find the value of f at x, we start at x, go up to the curve, and then over to the y-axis.

(b) The graph of f is also the graph of f^{-1}. To find the x that gave y, we start at y and go over to the curve and down to the x-axis. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.

(c) To draw the graph of f^{-1} in the usual way, we reflect the system across the line $y = x$.

(d) Then we interchange the letters x and y. We now have a normal-looking graph of f^{-1} as a function of x.

Figure 1.35 The graph of $y = f^{-1}(x)$.

The fact that the graphs of f and f^{-1} are reflections of each other across the line $y = x$ is to be expected because the input-output pairs (a, b) of f have been reversed to produce the input-output pairs (b, a) of f^{-1}.

The pictures in Figure 1.35 tell us how to express f^{-1} as a function of x algebraically.

Writing f^{-1} as a Function of x

1. Solve the equation $y = f(x)$ for x in terms of y.
2. Interchange x and y. The resulting formula will be $y = f^{-1}(x)$.

EXAMPLE 2 Finding the Inverse Function

Show that the function $y = f(x) = -2x + 4$ is one-to-one and find its inverse function.

SOLUTION

Every horizontal line intersects the graph of f exactly once, so f is one-to-one and has an inverse.

Step 1:

Solve for x in terms of y: $y = -2x + 4$

$$x = -\frac{1}{2}y + 2$$

continued

Step 2:

Interchange x and y: $y = -\frac{1}{2}x + 2$

The inverse of the function $f(x) = -2x + 4$ is the function $f^{-1}(x) = -(1/2)x + 2$. We can verify that both composites are the identity function.

$$f^{-1}(f(x)) = -\frac{1}{2}(-2x + 4) + 2 = x - 2 + 2 = x$$

$$f(f^{-1}(x)) = -2\left(-\frac{1}{2}x + 2\right) + 4 = x - 4 + 4 = x$$

Now try Exercise 13.

Graphing $y = f(x)$ and $y = f^{-1}(x)$ Parametrically

We can graph any function $y = f(x)$ as

$$x_1 = t, \quad y_1 = f(t).$$

Interchanging t and $f(t)$ produces parametric equations for the inverse:

$$x_2 = f(t), \quad y_2 = t.$$

We can use parametric graphing to graph the inverse of a function without finding an explicit rule for the inverse, as illustrated in Example 3.

EXAMPLE 3 Graphing the Inverse Parametrically

(a) Graph the one-to-one function $f(x) = x^2$, $x \geq 0$, together with its inverse and the line $y = x$, $x \geq 0$.

(b) Express the inverse of f as a function of x.

SOLUTION

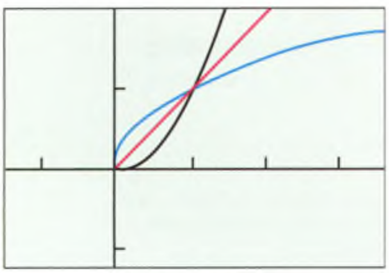
(a) We can graph the three functions parametrically as follows:

Graph of f: $x_1 = t, \quad y_1 = t^2, \quad t \geq 0$

Graph of f^{-1}: $x_2 = t^2, \quad y_2 = t$

Graph of $y = x$: $x_3 = t, \quad y_3 = t$

Figure 1.36 shows the three graphs.



[–1.5, 3.5] by [–1, 2]

Figure 1.36 The graphs of f and f^{-1} are reflections of each other across the line $y = x$. (Example 3)

(b) Next we find a formula for $f^{-1}(x)$.

Step 1:

Solve for x in terms of y.

$$y = x^2$$

$$\sqrt{y} = \sqrt{x^2}$$

$$\sqrt{y} = x \qquad \text{Because } x \geq 0.$$

Step 2:

Interchange x and y.

$$\sqrt{x} = y$$

Thus, $f^{-1}(x) = \sqrt{x}$.

Now try Exercise 27.

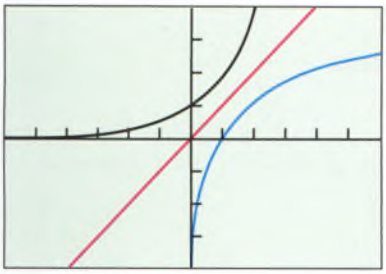
Logarithmic Functions

If a is any positive real number other than 1, the base a exponential function $f(x) = a^x$ is one-to-one. It therefore has an inverse. Its inverse is called the *base a logarithm function.*

DEFINITION Base *a* Logarithm Function

The **base *a* logarithm function** $y = \log_a x$ is the inverse of the base a exponential function $y = a^x$ $(a > 0, a \neq 1)$.

The domain of $\log_a x$ is $(0, \infty)$, the range of a^x. The range of $\log_a x$ is $(-\infty, \infty)$, the domain of a^x.



[−6, 6] by [−4, 4]

Figure 1.37 The graphs of $y = 2^x$ ($x_1 = t$, $y_1 = 2^t$), its inverse $y = \log_2 x$ ($x_2 = 2^t$, $y_2 = t$), and $y = x$ ($x_3 = t$, $y_3 = t$).

Because we have no technique for solving for x in terms of y in the equation $y = a^x$, we do not have an explicit formula for the logarithm function as a function of x. However, the graph of $y = \log_a x$ can be obtained by reflecting the graph of $y = a^x$ across the line $y = x$, or by using parametric graphing (Figure 1.37).

Logarithms with base e and base 10 are so important in applications that calculators have special keys for them. They also have their own special notation and names:

$$\log_e x = \ln x,$$

$$\log_{10} x = \log x.$$

The function $y = \ln x$ is called the **natural logarithm function** and $y = \log x$ is often called the **common logarithm function.**

Properties of Logarithms

Because a^x and $\log_a x$ are inverses of each other, composing them in either order gives the identity function. This gives two useful properties.

Inverse Properties for a^x and $\log_a x$

1. Base a: $a^{\log_a x} = x$, $\log_a a^x = x$, $a > 1, x > 0$
2. Base e: $e^{\ln x} = x$, $\ln e^x = x$, $x > 0$

These properties help us with the solution of equations that contain logarithms and exponential functions.

EXAMPLE 4 Using the Inverse Properties

Solve for x: **(a)** $\ln x = 3t + 5$ **(b)** $e^{2x} = 10$

SOLUTION

(a) $\ln x = 3t + 5$

$e^{\ln x} = e^{3t+5}$ Exponentiate both sides.

$x = e^{3t+5}$ Inverse Property

(b) $e^{2x} = 10$

$\ln e^{2x} = \ln 10$ Take logarithms of both sides.

$2x = \ln 10$ Inverse Property

$x = \frac{1}{2}\ln 10 \approx 1.15$

Now try Exercises 33 and 37.

The logarithm function has the following useful arithmetic properties.

Properties of Logarithms

For any real numbers $x > 0$ and $y > 0$,

1. *Product Rule*: $\log_a xy = \log_a x + \log_a y$
2. *Quotient Rule*: $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. *Power Rule*: $\log_a x^y = y \log_a x$

EXPLORATION 2 Supporting the Product Rule

Let $y_1 = \ln(ax)$, $y_2 = \ln x$, and $y_3 = y_1 - y_2$.

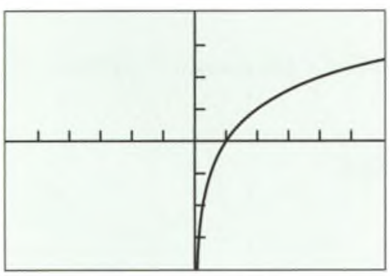
1. Graph y_1 and y_2 for $a = 2, 3, 4,$ and 5. How do the graphs of y_1 and y_2 appear to be related?
2. Support your finding by graphing y_3.
3. Confirm your finding algebraically.

The following formula allows us to evaluate $\log_a x$ for any base $a > 0$, $a \neq 1$, and to obtain its graph using the natural logarithm function on our grapher.

Change of Base Formula

$$\log_a x = \frac{\ln x}{\ln a}$$

$y = \dfrac{\ln x}{\ln 2}$



[–6, 6] by [–4, 4]

Figure 1.38 The graph of $f(x) = \log_2 x$ using $f(x) = (\ln x)/(\ln 2)$. (Example 5)

EXAMPLE 5 Graphing a Base *a* Logarithm Function

Graph $f(x) = \log_2 x$.

SOLUTION

We use the change of base formula to rewrite $f(x)$.

$$f(x) = \log_2 x = \frac{\ln x}{\ln 2}.$$

Figure 1.38 gives the graph of f. *Now try Exercise 41.*

Applications

In Section 1.3 we used graphical methods to solve exponential growth and decay problems. Now we can use the properties of logarithms to solve the same problems algebraically.

EXAMPLE 6 Finding Time

Sarah invests \$1000 in an account that earns 5.25% interest compounded annually. How long will it take the account to reach \$2500?

SOLUTION

Model The amount in the account at any time t in years is $1000(1.0525)^t$, so we need to solve the equation

$$1000(1.0525)^t = 2500.$$

continued

Table 1.15 Saudi Arabia's Natural Gas Production

Year	Cubic Feet (trillions)
1997	1.60
1998	1.65
1999	1.63
2000	1.76
2001	1.90

Source: Statistical Abstract of the United States, 2004–2005.

Solve Algebraically

$$(1.0525)^t = 2.5 \qquad \text{Divide by 1000.}$$
$$\ln (1.0525)^t = \ln 2.5 \qquad \text{Take logarithms of both sides.}$$
$$t \ln 1.0525 = \ln 2.5 \qquad \text{Power Rule}$$
$$t = \frac{\ln 2.5}{\ln 1.0525} \approx 17.9$$

Interpret The amount in Sarah's account will be \$2500 in about 17.9 years, or about 17 years and 11 months. ***Now try Exercise 47.***

EXAMPLE 7 Estimating Natural Gas Production

Table 1.15 shows the annual number of cubic feet in trillions of natural gas produced by Saudi Arabia for several years.

Find the natural logarithm regression equation for the data in Table 1.15 and use it to estimate the number of cubic feet of natural gas produced by Saudi Arabia in 2002. Compare with the actual amount of 2.00 trillion cubic feet in 2002.

SOLUTION

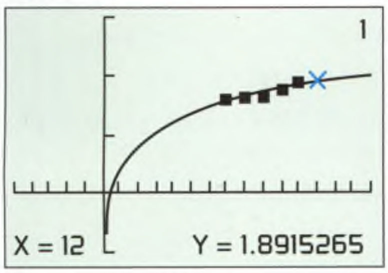
Model We let $x = 0$ represent 1990, $x = 1$ represent 1991, and so forth. We compute the natural logarithm regression equation to be

$$f(x) = 0.3730 + (0.611)\ln(x).$$

Solve Graphically Figure 1.39 shows the graph of f superimposed on the scatter plot of the data. The year 2002 is represented by $x = 12$. Reading from the graph we find $f(12) = 1.89$ trillion cubic feet.

Interpret The natural logarithmic model gives an underestimate of 0.11 trillion cubic feet of the 2002 natural gas production. ***Now try Exercise 49.***

$f(x) = 0.3730 + (0.611) \ln x$



[–5, 15] by [–1, 3]

Figure 1.39 The value of f at $x = 12$ is about 1.89. (Example 7)

Quick Review 1.5 *(For help, go to Sections 1.2, 1.3, and 1.4.)*

In Exercises 1–4, let $f(x) = \sqrt[3]{x-1}$, $g(x) = x^2 + 1$, and evaluate the expression.

1. $(f \circ g)(1)$

2. $(g \circ f)(-7)$

3. $(f \circ g)(x)$

4. $(g \circ f)(x)$

In Exercises 5 and 6, choose parametric equations and a parameter interval to represent the function on the interval specified.

5. $y = \dfrac{1}{x-1}, \quad x \geq 2$

6. $y = x, \quad x < -3$

In Exercises 7–10, find the points of intersection of the two curves. Round your answers to 2 decimal places.

7. $y = 2x - 3, \quad y = 5$

8. $y = -3x + 5, \quad y = -3$

9. **(a)** $y = 2^x, \quad y = 3$

(b) $y = 2^x, \quad y = -1$

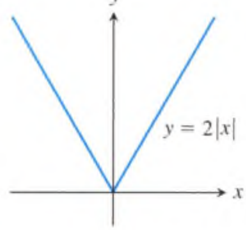
10. **(a)** $y = e^{-x}, \quad y = 4$

(b) $y = e^{-x}, \quad y = -1$

Section 1.5 Exercises

In Exercises 1–6, determine whether the function is one-to-one.

1.



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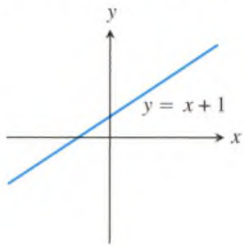

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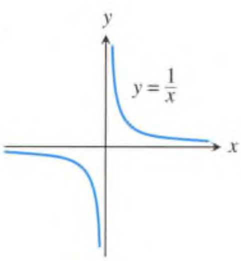

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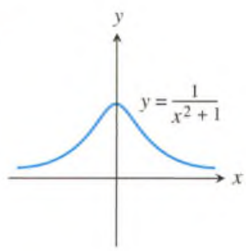
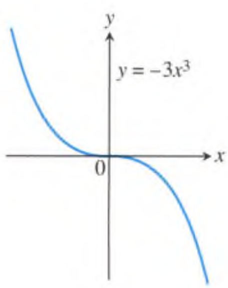
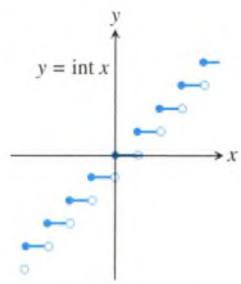


5.



6.



In Exercises 7–12, determine whether the function has an inverse function.

7. $y = \dfrac{3}{x-2} - 1$ **8.** $y = x^2 + 5x$ **9.** $y = x^3 - 4x + 6$

10. $y = x^3 + x$ **11.** $y = \ln x^2$ **12.** $y = 2^{3-x}$

In Exercises 13–24, find f^{-1} and verify that

$$(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x.$$

13. $f(x) = 2x + 3$ **14.** $f(x) = 5 - 4x$

15. $f(x) = x^3 - 1$ **16.** $f(x) = x^2 + 1, \quad x \geq 0$

17. $f(x) = x^2, \quad x \leq 0$ **18.** $f(x) = x^{2/3}, \quad x \geq 0$

19. $f(x) = -(x-2)^2, \quad x \leq 2$

20. $f(x) = x^2 + 2x + 1, \quad x \geq -1$

21. $f(x) = \dfrac{1}{x^2}, \quad x > 0$ **22.** $f(x) = \dfrac{1}{x^3}$

23. $f(x) = \dfrac{2x+1}{x+3}$ **24.** $f(x) = \dfrac{x+3}{x-2}$

In Exercises 25–32, use parametric graphing to graph f, f^{-1}, and $y = x$.

25. $f(x) = e^x$ **26.** $f(x) = 3^x$ **27.** $f(x) = 2^{-x}$

28. $f(x) = 3^{-x}$ **29.** $f(x) = \ln x$ **30.** $f(x) = \log x$

31. $f(x) = \sin^{-1} x$ **32.** $f(x) = \tan^{-1} x$

In Exercises 33–36, solve the equation algebraically. Support your solution graphically.

33. $(1.045)^t = 2$ **34.** $e^{0.05t} = 3$

35. $e^x + e^{-x} = 3$ **36.** $2^x + 2^{-x} = 5$

In Exercises 37 and 38, solve for y.

37. $\ln y = 2t + 4$ **38.** $\ln(y-1) - \ln 2 = x + \ln x$

In Exercises 39–42, draw the graph and determine the domain and range of the function.

39. $y = 2\ln(3-x) - 4$ **40.** $y = -3\log(x+2) + 1$

41. $y = \log_2(x+1)$ **42.** $y = \log_3(x-4)$

In Exercises 43 and 44, find a formula for f^{-1} and verify that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.

43. $f(x) = \dfrac{100}{1 + 2^{-x}}$ **44.** $f(x) = \dfrac{50}{1 + 1.1^{-x}}$

45. ***Self-inverse*** Prove that the function f is its own inverse.

(a) $f(x) = \sqrt{1 - x^2}, \quad x \geq 0$ **(b)** $f(x) = 1/x$

46. ***Radioactive Decay*** The half-life of a certain radioactive substance is 12 hours. There are 8 grams present initially.

(a) Express the amount of substance remaining as a function of time t.

(b) When will there be 1 gram remaining?

47. ***Doubling Your Money*** Determine how much time is required for a \$500 investment to double in value if interest is earned at the rate of 4.75% compounded annually.

48. ***Population Growth*** The population of Glenbrook is 375,000 and is increasing at the rate of 2.25% per year. Predict when the population will be 1 million.

In Exercises 49 and 50, let $x = 0$ represent 1990, $x = 1$ represent 1991, and so forth.

49. ***Natural Gas Production***

(a) Find a natural logarithm regression equation for the data in Table 1.16 and superimpose its graph on a scatter plot of the data.

Table 1.16 Canada's Natural Gas Production

Year	Cubic Feet (trillions)
1997	5.76
1998	5.98
1999	6.26
2000	6.47
2001	6.60

Source: Statistical Abstract of the United States, 2004–2005.

(b) Estimate the number of cubic feet of natural gas produced by Canada in 2002. Compare with the actual amount of 6.63 trillion cubic feet in 2002.

(c) Predict when Canadian natural gas production will reach 7 trillion cubic feet.

50. ***Natural Gas Production***

(a) Find a natural logarithm regression equation for the data in Table 1.17 and superimpose its graph on a scatter plot of the data.

Table 1.17 China's Natural Gas Production

Year	Cubic Feet (trillions)
1997	0.75
1998	0.78
1999	0.85
2000	0.96
2001	1.07

Source: Statistical Abstract of the United States, 2004–2005.

(b) Estimate the number of cubic feet of natural gas produced by China in 2002. Compare with the actual amount of 1.15 trillion cubic feet in 2002.

(c) Predict when China's natural gas production will reach 1.5 trillion cubic feet.

51. Group Activity ***Inverse Functions*** Let $y = f(x) = mx + b$, $m \neq 0$.

(a) Writing to Learn Give a convincing argument that f is a one-to-one function.

(b) Find a formula for the inverse of f. How are the slopes of f and f^{-1} related?

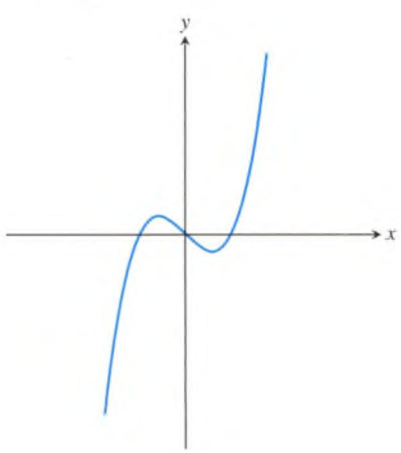
(c) If the graphs of two functions are parallel lines with a nonzero slope, what can you say about the graphs of the inverses of the functions?

(d) If the graphs of two functions are perpendicular lines with a nonzero slope, what can you say about the graphs of the inverses of the functions?

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

52. True or False The function displayed in the graph below is one-to-one. Justify your answer.



53. True or False If $(f \circ g)(x) = x$, then g is the inverse function of f. Justify your answer.

In Exercises 54 and 55, use the function $f(x) = 3 - \ln(x + 2)$.

54. Multiple Choice Which of the following is the domain of f?

(A) $x \neq -2$ **(B)** $(-\infty, \infty)$ **(C)** $(-2, \infty)$

(D) $[-1.9, \infty)$ **(E)** $(0, \infty)$

55. Multiple Choice Which of the following is the range of f?

(A) $(-\infty, \infty)$ **(B)** $(-\infty, 0)$ **(C)** $(-2, \infty)$

(D) $(0, \infty)$ **(E)** $(0, 5.3)$

56. Multiple Choice Which of the following is the inverse of $f(x) = 3x - 2$?

(A) $g(x) = \dfrac{1}{3x - 2}$ **(B)** $g(x) = x$ **(C)** $g(x) = 3x - 2$

(D) $g(x) = \dfrac{x - 2}{3}$ **(E)** $g(x) = \dfrac{x + 2}{3}$

57. Multiple Choice Which of the following is a solution of the equation $2 - 3^{-x} = -1$?

(A) $x = -2$ **(B)** $x = -1$ **(C)** $x = 0$

(D) $x = 1$ **(E)** There are no solutions.

Exploration

58. ***Supporting the Quotient Rule*** Let $y_1 = \ln(x/a)$, $y_2 = \ln x$, $y_3 = y_2 - y_1$, and $y_4 = e^{y_3}$.

(a) Graph y_1 and y_2 for $a = 2, 3, 4,$ and 5. How are the graphs of y_1 and y_2 related?

(b) Graph y_3 for $a = 2, 3, 4,$ and 5. Describe the graphs.

(c) Graph y_4 for $a = 2, 3, 4,$ and 5. Compare the graphs to the graph of $y = a$.

(d) Use $e^{y_3} = e^{y_2 - y_1} = a$ to solve for y_1.

Extending the Ideas

59. ***One-to-One Functions*** If f is a one-to-one function, prove that $g(x) = -f(x)$ is also one-to-one.

60. ***One-to-One Functions*** If f is a one-to-one function and $f(x)$ is never zero, prove that $g(x) = 1/f(x)$ is also one-to-one.

61. ***Domain and Range*** Suppose that $a \neq 0$, $b \neq 1$, and $b > 0$. Determine the domain and range of the function.

(a) $y = a(b^{c-x}) + d$ **(b)** $y = a \log_b (x - c) + d$

62. Group Activity ***Inverse Functions***

Let $f(x) = \dfrac{ax + b}{cx + d}$, $c \neq 0$, $ad - bc \neq 0$.

(a) Writing to Learn Give a convincing argument that f is one-to-one.

(b) Find a formula for the inverse of f.

(c) Find the horizontal and vertical asymptotes of f.

(d) Find the horizontal and vertical asymptotes of f^{-1}. How are they related to those of f?

1.6 Trigonometric Functions

What you'll learn about

- Radian Measure
- Graphs of Trigonometric Functions
- Periodicity
- Even and Odd Trigonometric Functions
- Transformations of Trigonometric Graphs
- Inverse Trigonometric Functions

. . . and why

Trigonometric functions can be used to model periodic behavior and applications such as musical notes.

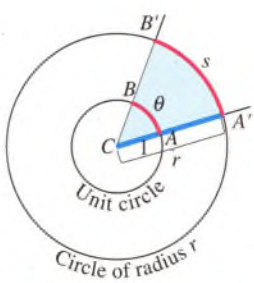


Figure 1.40 The radian measure of angle ACB is the length θ of arc AB on the unit circle centered at C. The value of θ can be found from any other circle, however, as the ratio s/r.

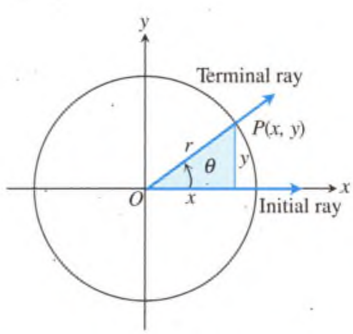


Figure 1.41 An angle θ in standard position.

Radian Measure

The **radian measure** of the angle ACB at the center of the unit circle (Figure 1.40) equals the length of the arc that ACB cuts from the unit circle.

EXAMPLE 1 Finding Arc Length

Find the length of an arc subtended on a circle of radius 3 by a central angle of measure $2\pi/3$.

SOLUTION

According to Figure 1.40, if s is the length of the arc, then

$$s = r\theta = 3(2\pi/3) = 2\pi.$$

Now try Exercise 1.

When an angle of measure θ is placed in *standard position* at the center of a circle of radius r (Figure 1.41), the six basic trigonometric functions of θ are defined as follows:

$$\textbf{sine: } \sin\theta = \frac{y}{r} \qquad \textbf{cosecant: } \csc\theta = \frac{r}{y}$$

$$\textbf{cosine: } \cos\theta = \frac{x}{r} \qquad \textbf{secant: } \sec\theta = \frac{r}{x}$$

$$\textbf{tangent: } \tan\theta = \frac{y}{x} \qquad \textbf{cotangent: } \cot\theta = \frac{x}{y}$$

Graphs of Trigonometric Functions

When we graph trigonometric functions in the coordinate plane, we usually denote the independent variable (radians) by x instead of θ. Figure 1.42 on the next page shows sketches of the six trigonometric functions. It is a good exercise for you to compare these with what you see in a grapher viewing window. (Some graphers have a "trig viewing window.")

EXPLORATION 1 Unwrapping Trigonometric Functions

Set your grapher in *radian mode,* parametric mode, and *simultaneous mode* (all three). Enter the parametric equations

$$x_1 = \cos t, \quad y_1 = \sin t \quad \text{and} \quad x_2 = t, \quad y_2 = \sin t.$$

1. Graph for $0 \le t \le 2\pi$ in the window $[-1.5, 2\pi]$ by $[-2.5, 2.5]$. Describe the two curves. (You may wish to make the viewing window square.)
2. Use trace to compare the y-values of the two curves.
3. Repeat part 2 in the window $[-1.5, 4\pi]$ by $[-5, 5]$, using the parameter interval $0 \le t \le 4\pi$.
4. Let $y_2 = \cos t$. Use trace to compare the x-values of curve 1 (the unit circle) with the y-values of curve 2 using the parameter intervals $[0, 2\pi]$ and $[0, 4\pi]$.
5. Set $y_2 = \tan t$, $\csc t$, $\sec t$, and $\cot t$. Graph each in the window $[-1.5, 2\pi]$ by $[-2.5, 2.5]$ using the interval $0 \le t \le 2\pi$. How is a y-value of curve 2 related to the corresponding point on curve 1? (Use trace to explore the curves.)

Angle Convention: Use Radians

From now on in this book it is assumed that all angles are measured in radians unless degrees or some other unit is stated explicitly. When we talk about the angle $\pi/3$, we mean $\pi/3$ radians (which is 60°), not $\pi/3$ degrees. When you do calculus, keep your calculator in radian mode.

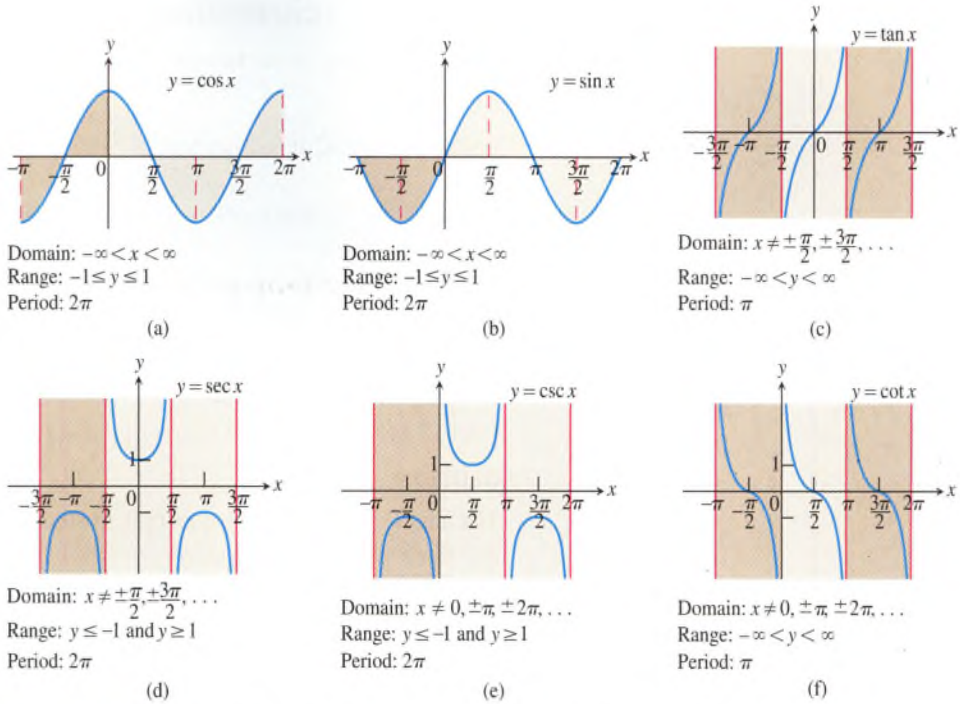


Figure 1.42 Graphs of the (a) cosine, (b) sine, (c) tangent, (d) secant, (e) cosecant, and (f) cotangent functions using radian measure.

Periods of Trigonometric Functions

Period π: $\tan(x + \pi) = \tan x$
$\cot(x + \pi) = \cot x$

Period 2π: $\sin(x + 2\pi) = \sin x$
$\cos(x + 2\pi) = \cos x$
$\sec(x + 2\pi) = \sec x$
$\csc(x + 2\pi) = \csc x$

Periodicity

When an angle of measure θ and an angle of measure $\theta + 2\pi$ are in standard position, their terminal rays coincide. The two angles therefore have the same trigonometric function values:

$$\begin{aligned} &\cos(\theta + 2\pi) = \cos\theta \quad \sin(\theta + 2\pi) = \sin\theta \quad \tan(\theta + 2\pi) = \tan\theta \\ &\sec(\theta + 2\pi) = \sec\theta \quad \csc(\theta + 2\pi) = \csc\theta \quad \cot(\theta + 2\pi) = \cot\theta \end{aligned} \tag{1}$$

Similarly, $\cos(\theta - 2\pi) = \cos\theta$, $\sin(\theta - 2\pi) = \sin\theta$, and so on.

We see the values of the trigonometric functions repeat at regular intervals. We describe this behavior by saying that the six basic trigonometric functions are *periodic*.

> **DEFINITION Periodic Function, Period**
>
> A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

As we can see in Figure 1.42, the functions $\cos x$, $\sin x$, $\sec x$, and $\csc x$ are periodic with period 2π. The functions $\tan x$ and $\cot x$ are periodic with period π.

Even and Odd Trigonometric Functions

The graphs in Figure 1.42 suggest that $\cos x$ and $\sec x$ are even functions because their graphs are symmetric about the y-axis. The other four basic trigonometric functions are odd.

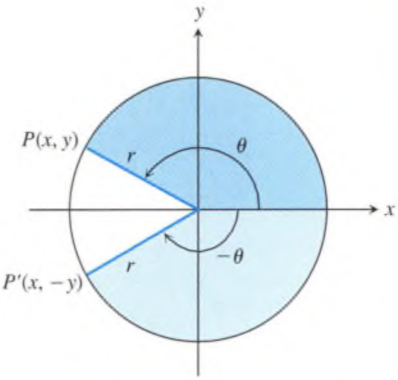


Figure 1.43 Angles of opposite sign. (Example 2)

EXAMPLE 2 Confirming Even and Odd

Show that cosine is an even function and sine is odd.

SOLUTION

From Figure 1.43 it follows that

$$\cos(-\theta) = \frac{x}{r} = \cos\theta, \quad \sin(-\theta) = \frac{-y}{r} = -\sin\theta,$$

so cosine is an even function and sine is odd. ***Now try Exercise 5.***

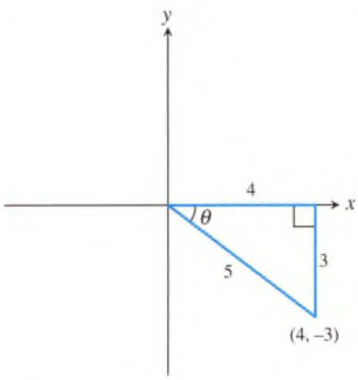


Figure 1.44 The angle θ in standard position. (Example 3)

EXAMPLE 3 Finding Trigonometric Values

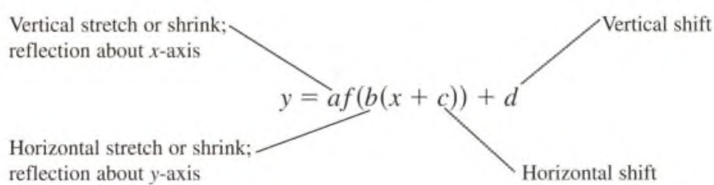
Find all the trigonometric values of θ if $\sin\theta = -3/5$ and $\tan\theta < 0$.

SOLUTION

The angle θ is in the fourth quadrant, as shown in Figure 1.44, because its sine and tangent are negative. From this figure we can read that $\cos\theta = 4/5$, $\tan\theta = -3/4$, $\csc\theta = -5/3$, $\sec\theta = 5/4$, and $\cot\theta = -4/3$. ***Now try Exercise 9.***

Transformations of Trigonometric Graphs

The rules for shifting, stretching, shrinking, and reflecting the graph of a function apply to the trigonometric functions. The following diagram will remind you of the controlling parameters.



The general sine function or **sinusoid** can be written in the form

$$f(x) = A\sin\left[\frac{2\pi}{B}(x - C)\right] + D,$$

where $|A|$ is the *amplitude,* $|B|$ is the *period,* C is the *horizontal shift,* and D is the *vertical shift.*

EXAMPLE 4 Graphing a Trigonometric Function

Determine the **(a)** period, **(b)** domain, **(c)** range, and **(d)** draw the graph of the function $y = 3\cos(2x - \pi) + 1$.

SOLUTION

We can rewrite the function in the form

$$y = 3\cos\left[2\left(x - \frac{\pi}{2}\right)\right] + 1.$$

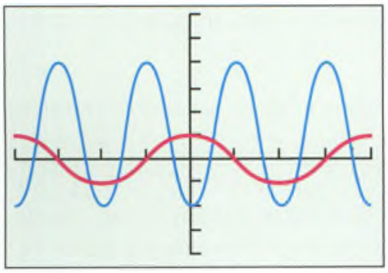
(a) The period is given by $2\pi/B$, where $2\pi/B = 2$. The period is π.

(b) The domain is $(-\infty, \infty)$.

(c) The graph is a basic cosine curve with amplitude 3 that has been shifted up 1 unit. Thus, the range is $[-2, 4]$.

continued

$y = 3\cos(2x - \pi) + 1,\ y = \cos x$



$[-2\pi, 2\pi]$ by $[-4, 6]$

Figure 1.45 The graph of $y = 3\cos(2x - \pi) + 1$ (blue) and the graph of $y = \cos x$ (red). (Example 4)

(d) The graph has been shifted to the right $\pi/2$ units. The graph is shown in Figure 1.45 together with the graph of $y = \cos x$. Notice that four periods of $y = 3\cos(2x - \pi) + 1$ are drawn in this window. ***Now try Exercise 13.***

Musical notes are pressure waves in the air. The wave behavior can be modeled with great accuracy by general sine curves. Devices called Calculator Based Laboratory™ (CBL) systems can record these waves with the aid of a microphone. The data in Table 1.18 give pressure displacement versus time in seconds of a musical note produced by a tuning fork and recorded with a CBL system.

Table 1.18 Tuning Fork Data

Time	Pressure	Time	Pressure	Time	Pressure
0.00091	−0.080	0.00271	−0.141	0.00453	0.749
0.00108	0.200	0.00289	−0.309	0.00471	0.581
0.00125	0.480	0.00307	−0.348	0.00489	0.346
0.00144	0.693	0.00325	−0.248	0.00507	0.077
0.00162	0.816	0.00344	−0.041	0.00525	−0.164
0.00180	0.844	0.00362	0.217	0.00543	−0.320
0.00198	0.771	0.00379	0.480	0.00562	−0.354
0.00216	0.603	0.00398	0.681	0.00579	−0.248
0.00234	0.368	0.00416	0.810	0.00598	−0.035
0.00253	0.099	0.00435	0.827		

EXAMPLE 5 Finding the Frequency of a Musical Note

Consider the tuning fork data in Table 1.18.

(a) Find a sinusoidal regression equation (general sine curve) for the data and superimpose its graph on a scatter plot of the data.

(b) The *frequency* of a musical note, or wave, is measured in cycles per second, or hertz (1 Hz = 1 cycle per second). The frequency is the reciprocal of the *period* of the wave, which is measured in seconds per cycle. Estimate the frequency of the note produced by the tuning fork.

SOLUTION

(a) The sinusoidal regression equation produced by our calculator is approximately

$$y = 0.6\sin(2488.6x - 2.832) + 0.266.$$

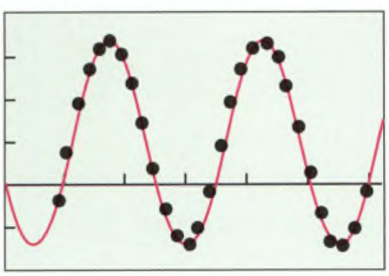
Figure 1.46 shows its graph together with a scatter plot of the tuning fork data.

(b) The period is $\dfrac{2\pi}{2488.6}$ sec, so the frequency is $\dfrac{2488.6}{2\pi} \approx 396$ Hz.

Interpretation The tuning fork is vibrating at a frequency of about 396 Hz. On the pure tone scale, this is the note G above middle C. It is a few cycles per second different from the frequency of the G we hear on a piano's tempered scale, 392 Hz.

Now try Exercise 23.

$y = 0.6\sin(2488.6x - 2.832) + 0.266$



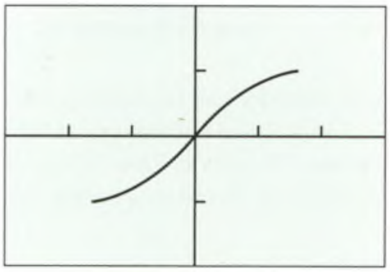
$[0, 0.0062]$ by $[-0.5, 1]$

Figure 1.46 A sinusoidal regression model for the tuning fork data in Table 1.18. (Example 5)

Inverse Trigonometric Functions

None of the six basic trigonometric functions graphed in Figure 1.42 is one-to-one. These functions do not have inverses. However, in each case the domain can be restricted to produce a new function that does have an inverse, as illustrated in Example 6.

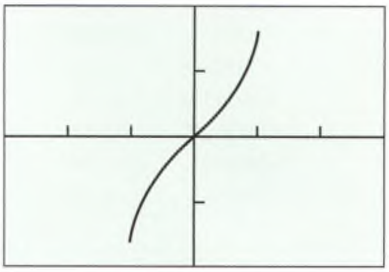
$x = t, y = \sin t, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$



[–3, 3] by [–2, 2]

(a)

$x = \sin t, y = t, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$



[–3, 3] by [–2, 2]

(b)

Figure 1.47 (a) A restricted sine function and (b) its inverse. (Example 6)

EXAMPLE 6 Restricting the Domain of the Sine

Show that the function $y = \sin x$, $-\pi/2 \leq x \leq \pi/2$, is one-to-one, and graph its inverse.

SOLUTION

Figure 1.47a shows the graph of this restricted sine function using the parametric equations

$$x_1 = t, \quad y_1 = \sin t, \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}.$$

This restricted sine function is one-to-one because it does not repeat any output values. It therefore has an inverse, which we graph in Figure 1.47b by interchanging the ordered pairs using the parametric equations

$$x_2 = \sin t, \quad y_2 = t, \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}.$$

Now try Exercise 25.

The inverse of the restricted sine function of Example 6 is called the *inverse sine function.* The inverse sine of x is the angle whose sine is x. It is denoted by $\sin^{-1} x$ or $\arcsin x$. Either notation is read "arcsine of x" or "the inverse sine of x."

The domains of the other basic trigonometric functions can also be restricted to produce a function with an inverse. The domains and ranges of the resulting inverse functions become parts of their definitions.

DEFINITIONS Inverse Trigonometric Functions

Function	Domain	Range
$y = \cos^{-1} x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$y = \sin^{-1} x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$y = \tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \sec^{-1} x$	$\|x\| \geq 1$	$0 \leq y \leq \pi, y \neq \frac{\pi}{2}$
$y = \csc^{-1} x$	$\|x\| \geq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$
$y = \cot^{-1} x$	$-\infty < x < \infty$	$0 < y < \pi$

The graphs of the six inverse trigonometric functions are shown in Figure 1.48.

EXAMPLE 7 Finding Angles in Degrees and Radians

Find the measure of $\cos^{-1}(-0.5)$ in degrees and radians.

SOLUTION

Put the calculator in degree mode and enter $\cos^{-1}(-0.5)$. The calculator returns 120, which means 120 degrees. Now put the calculator in radian mode and enter $\cos^{-1}(-0.5)$. The calculator returns 2.094395102, which is the measure of the angle in radians. You can check that $2\pi/3 \approx 2.094395102$.

Now try Exercise 27.

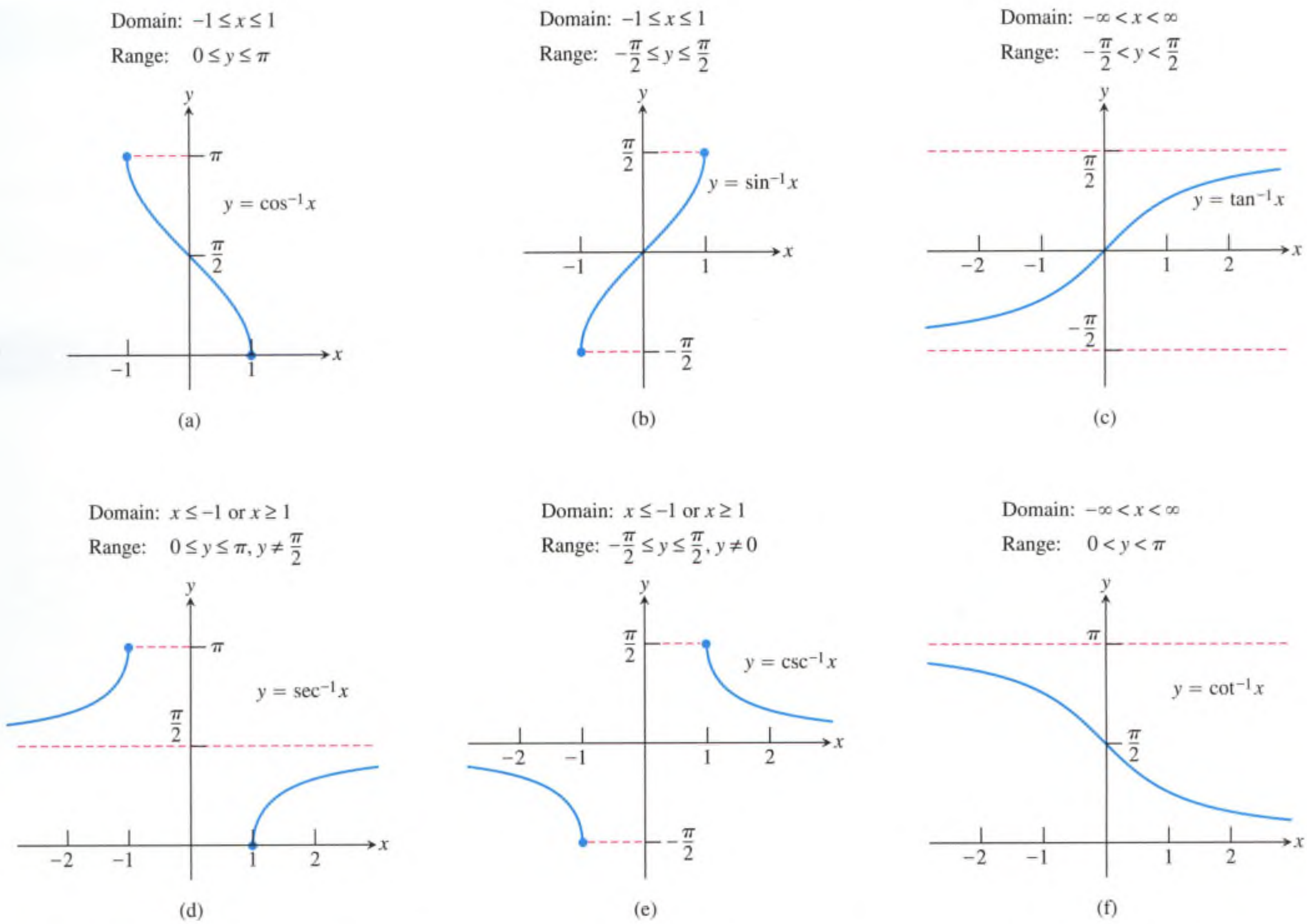


Figure 1.48 Graphs of (a) $y = \cos^{-1} x$, (b) $y = \sin^{-1} x$, (c) $y = \tan^{-1} x$, (d) $y = \sec^{-1} x$, (e) $y = \csc^{-1} x$, and (f) $y = \cot^{-1} x$.

EXAMPLE 8 Using the Inverse Trigonometric Functions

Solve for x.

(a) $\sin x = 0.7$ in $0 \le x < 2\pi$

(b) $\tan x = -2$ in $-\infty < x < \infty$

SOLUTION

(a) Notice that $x = \sin^{-1}(0.7) \approx 0.775$ is in the first quadrant, so 0.775 is one solution of this equation. The angle $\pi - x$ is in the second quadrant and has sine equal to 0.7. Thus two solutions in this interval are

$$\sin^{-1}(0.7) \approx 0.775 \quad \text{and} \quad \pi - \sin^{-1}(0.7) \approx 2.366.$$

(b) The angle $x = \tan^{-1}(-2) \approx -1.107$ is in the fourth quadrant and is the only solution to this equation in the interval $-\pi/2 < x < \pi/2$ where $\tan x$ is one-to-one. Since $\tan x$ is periodic with period π, the solutions to this equation are of the form

$$\tan^{-1}(-2) + k\pi \quad \approx -1.107 + k\pi$$

where k is any integer. ***Now try Exercise 31.***

Quick Review 1.6 *(For help, go to Sections 1.2 and 1.6.)*

In Exercises 1–4, convert from radians to degrees or degrees to radians.

1. $\pi/3$ **2.** -2.5 **3.** $-40°$ **4.** $45°$

In Exercises 5–7, solve the equation graphically in the given interval.

5. $\sin x = 0.6,\quad 0 \le x \le 2\pi$ **6.** $\cos x = -0.4,\quad 0 \le x \le 2\pi$

7. $\tan x = 1,\quad -\dfrac{\pi}{2} \le x < \dfrac{3\pi}{2}$

8. Show that $f(x) = 2x^2 - 3$ is an even function. Explain why its graph is symmetric about the y-axis.

9. Show that $f(x) = x^3 - 3x$ is an odd function. Explain why its graph is symmetric about the origin.

10. Give one way to restrict the domain of the function $f(x) = x^4 - 2$ to make the resulting function one-to-one.

Section 1.6 Exercises

In Exercises 1–4, the angle lies at the center of a circle and subtends an arc of the circle. Find the missing angle measure, circle radius, or arc length.

	Angle	Radius	Arc Length
1.	$5\pi/8$	2	?
2.	175°	?	10
3.	?	14	7
4.	?	6	$3\pi/2$

In Exercises 5–8, determine if the function is even or odd.

5. secant **6.** tangent

7. cosecant **8.** cotangent

In Exercises 9 and 10, find all the trigonometric values of θ with the given conditions.

9. $\cos\theta = -\dfrac{15}{17},\quad \sin\theta > 0$

10. $\tan\theta = -1,\quad \sin\theta < 0$

In Exercises 11–14, determine **(a)** the period, **(b)** the domain, **(c)** the range, and **(d)** draw the graph of the function.

11. $y = 3\csc(3x + \pi) - 2$ **12.** $y = 2\sin(4x + \pi) + 3$

13. $y = -3\tan(3x + \pi) + 2$

14. $y = 2\sin\left(2x + \dfrac{\pi}{3}\right)$

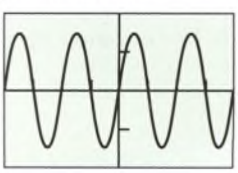
In Exercises 15 and 16, choose an appropriate viewing window to display two complete periods of each trigonometric function in radian mode.

15. **(a)** $y = \sec x$ **(b)** $y = \csc x$ **(c)** $y = \cot x$

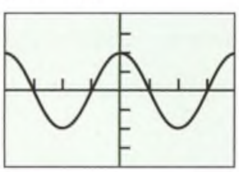
16. **(a)** $y = \sin x$ **(b)** $y = \cos x$ **(c)** $y = \tan x$

In Exercises 17–22, specify **(a)** the period, **(b)** the amplitude, and **(c)** identify the viewing window that is shown.

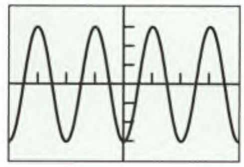
17. $y = 1.5\sin 2x$



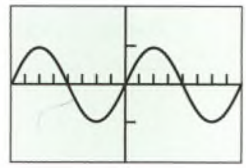
18. $y = 2\cos 3x$



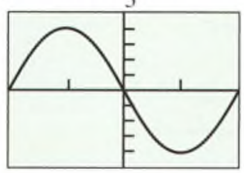
19. $y = -3\cos 2x$



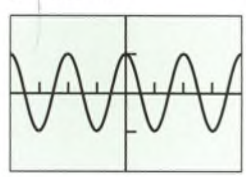
20. $y = 5\sin\dfrac{x}{2}$



21. $y = -4\sin\dfrac{\pi}{3}x$



22. $y = \cos\pi x$



23. Group Activity A musical note like that produced with a tuning fork or pitch meter is a pressure wave. Table 1.19 gives frequencies (in Hz) of musical notes on the tempered scale. The pressure versus time tuning fork data in Table 1.20 were collected using a CBL™ and a microphone.

Table 1.19 Frequencies of Notes

Note	Frequency (Hz)
C	262
C♯ or D♭	277
D	294
D♯ or E♭	311
E	330
F	349
F♯ or G♭	370
G	392
G♯ or A♭	415
A	440
A♯ or B♭	466
B	494
C (next octave)	524

Source: CBL™ System Experimental Workbook, Texas Instruments, Inc., 1994.

Table 1.20 **Tuning Fork Data**

Time (s)	Pressure	Time (s)	Pressure
0.0002368	1.29021	0.0049024	−1.06632
0.0005664	1.50851	0.0051520	0.09235
0.0008256	1.51971	0.0054112	1.44694
0.0010752	1.51411	0.0056608	1.51411
0.0013344	1.47493	0.0059200	1.51971
0.0015840	0.45619	0.0061696	1.51411
0.0018432	−0.89280	0.0064288	1.43015
0.0020928	−1.51412	0.0066784	0.19871
0.0023520	−1.15588	0.0069408	−1.06072
0.0026016	−0.04758	0.0071904	−1.51412
0.0028640	1.36858	0.0074496	−0.97116
0.0031136	1.50851	0.0076992	0.23229
0.0033728	1.51971	0.0079584	1.46933
0.0036224	1.51411	0.0082080	1.51411
0.0038816	1.45813	0.0084672	1.51971
0.0041312	0.32185	0.0087168	1.50851
0.0043904	−0.97676	0.0089792	1.36298
0.0046400	−1.51971		

(a) Find a sinusoidal regression equation for the data in Table 1.20 and superimpose its graph on a scatter plot of the data.

(b) Determine the frequency of and identify the musical note produced by the tuning fork.

24. ***Temperature Data*** Table 1.21 gives the average monthly temperatures for St. Louis for a 12-month period starting with January. Model the monthly temperature with an equation of the form

$$y = a \sin [b(t - h)] + k,$$

y in degrees Fahrenheit, t in months, as follows:

Table 1.21 **Temperature Data for St. Louis**

Time (months)	Temperature (°F)
1	34
2	30
3	39
4	44
5	58
6	67
7	78
8	80
9	72
10	63
11	51
12	40

(a) Find the value of b assuming that the period is 12 months.

(b) How is the amplitude a related to the difference $80° - 30°$?

(c) Use the information in (b) to find k.

(d) Find h, and write an equation for y.

(e) Superimpose a graph of y on a scatter plot of the data.

In Exercises 25–26, show that the function is one-to-one, and graph its inverse.

25. $y = \cos x, \quad 0 \le x \le \pi$

26. $y = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

In Exercises 27–30, give the measure of the angle in radians and degrees. Give exact answers whenever possible.

27. $\sin^{-1}(0.5)$

28. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

29. $\tan^{-1}(-5)$

30. $\cos^{-1}(0.7)$

In Exercises 31–36, solve the equation in the specified interval.

31. $\tan x = 2.5, \quad 0 \le x \le 2\pi$

32. $\cos x = -0.7, \quad 2\pi \le x < 4\pi$

33. $\csc x = 2, \quad 0 < x < 2\pi$

34. $\sec x = -3, \quad -\pi \le x < \pi$

35. $\sin x = -0.5, \quad -\infty < x < \infty$

36. $\cot x = -1, \quad -\infty < x < \infty$

In Exercises 37–40, use the given information to find the values of the six trigonometric functions at the angle θ. Give exact answers.

37. $\theta = \sin^{-1}\left(\frac{8}{17}\right)$

38. $\theta = \tan^{-1}\left(-\frac{5}{12}\right)$

39. The point $P(-3, 4)$ is on the terminal side of θ.

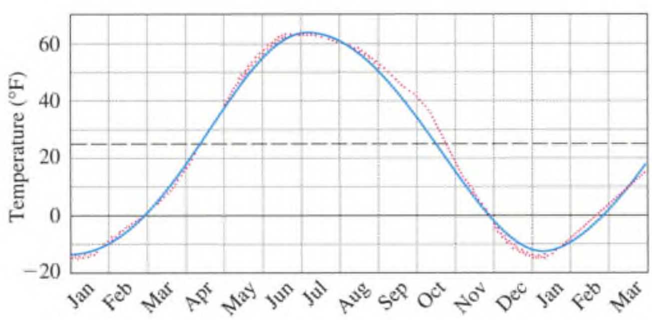
40. The point $P(-2, 2)$ is on the terminal side of θ.

In Exercises 41 and 42, evaluate the expression.

41. $\sin\left(\cos^{-1}\left(\frac{7}{11}\right)\right)$

42. $\tan\left(\sin^{-1}\left(\frac{9}{13}\right)\right)$

43. ***Temperatures in Fairbanks, Alaska*** Find the **(a)** amplitude, **(b)** period, **(c)** horizontal shift, and **(d)** vertical shift of the model used in the figure below. **(e)** Then write the equation for the model.



Normal mean air temperature for Fairbanks, Alaska, plotted as data points (red). The approximating sine function $f(x)$ is drawn in blue.
Source: "Is the Curve of Temperature Variation a Sine Curve?" by B. M. Lando and C. A. Lando, *The Mathematics Teacher, 7.6,* Fig. 2, p. 535 (Sept . 1977).

44. ***Temperatures in Fairbanks, Alaska*** Use the equation of Exercise 43 to approximate the answers to the following questions about the temperatures in Fairbanks, Alaska, shown in the figure in Exercise 43. Assume that the year has 365 days.

(a) What are the highest and lowest mean daily temperatures?

(b) What is the average of the highest and lowest mean daily temperatures? Why is this average the vertical shift of the function?

45. *Even-Odd*

(a) Show that $\cot x$ is an odd function of x.

(b) Show that the quotient of an even function and an odd function is an odd function.

46. *Even-Odd*

(a) Show that $\csc x$ is an odd function of x.

(b) Show that the reciprocal of an odd function is odd.

47. *Even-Odd* Show that the product of an even function and an odd function is an odd function.

48. *Finding the Period* Give a convincing argument that the period of $\tan x$ is π.

49. *Sinusoidal Regression* Table 1.22 gives the values of the function

$$f(x) = a \sin(bx + c) + d$$

accurate to two decimals.

Table 1.22 Values of a Function

x	$f(x)$
1	3.42
2	0.73
3	0.12
4	2.16
5	4.97
6	5.97

(a) Find a sinusoidal regression equation for the data.

(b) Rewrite the equation with a, b, c, and d rounded to the nearest integer.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

50. True or False The period of $y = \sin(x/2)$ is π. Justify your answer.

51. True or False The amplitude of $y = \frac{1}{2}\cos x$ is 1. Justify your answer.

In Exercises 52–54, $f(x) = 2\cos(4x + \pi) - 1$.

52. Multiple Choice Which of the following is the domain of f?

(A) $[-\pi, \pi]$ **(B)** $[-3, 1]$ **(C)** $[-1, 4]$

(D) $(-\infty, \infty)$ **(E)** $x \neq 0$

53. Multiple Choice Which of the following is the range of f?

(A) $(-3, 1)$ **(B)** $[-3, 1]$ **(C)** $(-1, 4)$

(D) $[-1, 4]$ **(E)** $(-\infty, \infty)$

54. Multiple Choice Which of the following is the period of f?

(A) 4π **(B)** 3π **(C)** 2π **(D)** π **(E)** $\pi/2$

55. Multiple Choice Which of the following is the measure of $\tan^{-1}(-\sqrt{3})$ in degrees?

(A) $-60°$ **(B)** $-30°$ **(C)** $30°$ **(D)** $60°$ **(E)** $120°$

Exploration

56. *Trigonometric Identities* Let $f(x) = \sin x + \cos x$.

(a) Graph $y = f(x)$. Describe the graph.

(b) Use the graph to identify the amplitude, period, horizontal shift, and vertical shift.

(c) Use the formula

$$\sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin(\alpha + \beta)$$

for the sine of the sum of two angles to confirm your answers.

Extending the Ideas

57. Exploration Let $y = \sin(ax) + \cos(ax)$.

Use the symbolic manipulator of a computer algebra system (CAS) to help you with the following:

(a) Express y as a sinusoid for $a = 2, 3, 4$, and 5.

(b) Conjecture another formula for y for a equal to any positive integer n.

(c) Check your conjecture with a CAS.

(d) Use the formula for the sine of the sum of two angles (see Exercise 56c) to confirm your conjecture.

58. Exploration Let $y = a\sin x + b\cos x$.

Use the symbolic manipulator of a computer algebra system (CAS) to help you with the following:

(a) Express y as a sinusoid for the following pairs of values: $a = 2, b = 1$; $a = 1, b = 2$; $a = 5, b = 2$; $a = 2, b = 5$; $a = 3, b = 4$.

(b) Conjecture another formula for y for any pair of positive integers. Try other values if necessary.

(c) Check your conjecture with a CAS.

(d) Use the following formulas for the sine or cosine of a sum or difference of two angles to confirm your conjecture.

$$\sin\alpha\cos\beta \pm \cos\alpha\sin\beta = \sin(\alpha \pm \beta)$$

$$\cos\alpha\cos\beta \pm \sin\alpha\sin\beta = \cos(\alpha \mp \beta)$$

In Exercises 59 and 60, show that the function is periodic and find its period.

59. $y = \sin^3 x$ **60.** $y = |\tan x|$

In Exercises 61 and 62, graph one period of the function.

61. $f(x) = \sin(60x)$

62. $f(x) = \cos(60\pi x)$

Quick Quiz for AP* Preparation: Sections 1.4–1.6

You should solve the following problems without using a graphing calculator.

1. **Multiple Choice** Which of the following is the domain of $f(x) = -\log_2(x + 3)$?
   (A) $(-\infty, \infty)$ **(B)** $(-\infty, 3)$ **(C)** $(-3, \infty)$
   (D) $[-3, \infty)$ **(E)** $(-\infty, 3]$

2. **Multiple Choice** Which of the following is the range of $f(x) = 5\cos(x + \pi) + 3$?
   (A) $(-\infty, \infty)$ **(B)** $[2, 4]$ **(C)** $[-8, 2]$
   (D) $[-2, 8]$ **(E)** $\left[-\frac{2}{5}, \frac{8}{5}\right]$

3. **Multiple Choice** Which of the following gives the solution of $\tan x = -1$ in $\pi < x < \frac{3\pi}{2}$?
   (A) $-\frac{\pi}{4}$ **(B)** $\frac{\pi}{4}$ **(C)** $\frac{\pi}{3}$ **(D)** $\frac{3\pi}{4}$ **(E)** $\frac{5\pi}{4}$

4. **Free Response** Let $f(x) = 5x - 3$.
   (a) Find the inverse g of f.
   (b) Compute $f \circ g(x)$. Show your work.
   (c) Compute $g \circ f(x)$. Show your work.

Chapter 1 Key Terms

absolute value function (p. 17)
base a logarithm function (p. 40)
boundary of an interval (p. 13)
boundary points (p. 13)
change of base formula (p. 42)
closed interval (p. 13)
common logarithm function (p. 41)
composing (p. 18)
composite function (p. 17)
compounded continuously (p. 25)
cosecant function (p. 46)
cosine function (p. 46)
cotangent function (p. 46)
dependent variable (p. 12)
domain (p. 12)
even function (p. 15)
exponential decay (p. 24)
exponential function base a (p. 22)
exponential growth (p. 24)
function (p. 12)
general linear equation (p. 5)
graph of a function (p. 13)
graph of a relation (p. 30)
grapher failure (p. 15)
half-life (p. 24)
half-open interval (p. 13)
identity function (p. 38)
increments (p. 3)
independent variable (p. 12)
initial point of parametrized curve (p. 30)
interior of an interval (p. 13)
interior points of an interval (p. 13)
inverse cosecant function (p. 50)
inverse cosine function (p. 50)
inverse cotangent function (p. 50)
inverse function (p. 38)
inverse properties for a^x and $\log_a x$ (p. 41)
inverse secant function (p. 50)
inverse sine function (p. 50)
inverse tangent function (p. 50)
linear regression (p. 7)
natural domain (p. 13)
natural logarithm function (p. 41)
odd function (p. 15)
one-to-one function (p. 37)
open interval (p. 13)
parallel lines (p. 4)
parameter (p. 30)
parameter interval (p. 30)
parametric curve (p. 30)
parametric equations (p. 30)
parametrization of a curve (p. 30)
parametrize (p. 30)
period of a function (p. 47)
periodic function (p. 47)
perpendicular lines (p. 4)
piecewise defined function (p. 16)
point-slope equation (p. 4)
power rule for logarithms (p. 41)
product rule for logarithms (p. 41)
quotient rule for logarithms (p. 41)
radian measure (p. 46)
range (p. 12)
regression analysis (p. 7)
regression curve (p. 7)
relation (p. 30)
rise (p. 3)
rules for exponents (p. 23)
run (p. 3)
scatter plot (p. 7)
secant function (p. 46)
sine function (p. 46)
sinusoid (p. 48)
sinusoidal regression (p. 49)
slope (p. 4)
slope-intercept equation (p. 5)
symmetry about the origin (p. 15)
symmetry about the y-axis (p. 15)
tangent function (p. 46)
terminal point of parametrized curve (p. 30)
witch of Agnesi (p. 33)
x-intercept (p. 5)
y-intercept (p. 5)

Chapter 1 Review Exercises

The collection of exercises marked in red could be used as a chapter test.

In Exercises 1–14, write an equation for the specified line.

1. through $(1, -6)$ with slope 3
2. through $(-1, 2)$ with slope $-1/2$
3. the vertical line through $(0, -3)$
4. through $(-3, 6)$ and $(1, -2)$
5. the horizontal line through $(0, 2)$
6. through $(3, 3)$ and $(-2, 5)$
7. with slope -3 and y-intercept 3
8. through $(3, 1)$ and parallel to $2x - y = -2$
9. through $(4, -12)$ and parallel to $4x + 3y = 12$
10. through $(-2, -3)$ and perpendicular to $3x - 5y = 1$
11. through $(-1, 2)$ and perpendicular to $\frac{1}{2}x + \frac{1}{3}y = 1$
12. with x-intercept 3 and y-intercept -5
13. the line $y = f(x)$, where f has the following values:

x	-2	2	4
$f(x)$	4	2	1

14. through $(4, -2)$ with x-intercept -3

In Exercises 15–18, determine whether the graph of the function is symmetric about the y-axis, the origin, or neither.

15. $y = x^{1/5}$
16. $y = x^{2/5}$
17. $y = x^2 - 2x - 1$
18. $y = e^{-x^2}$

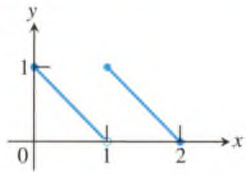
In Exercises 19–26, determine whether the function is even, odd, or neither.

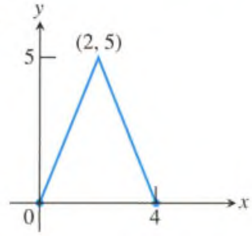
19. $y = x^2 + 1$
20. $y = x^5 - x^3 - x$
21. $y = 1 - \cos x$
22. $y = \sec x \tan x$
23. $y = \dfrac{x^4 + 1}{x^3 - 2x}$
24. $y = 1 - \sin x$
25. $y = x + \cos x$
26. $y = \sqrt{x^4 - 1}$

In Exercises 27–38, find the **(a)** domain and **(b)** range, and **(c)** graph the function.

27. $y = |x| - 2$
28. $y = -2 + \sqrt{1 - x}$
29. $y = \sqrt{16 - x^2}$
30. $y = 3^{2-x} + 1$
31. $y = 2e^{-x} - 3$
32. $y = \tan(2x - \pi)$
33. $y = 2\sin(3x + \pi) - 1$
34. $y = x^{2/5}$
35. $y = \ln(x - 3) + 1$
36. $y = -1 + \sqrt[3]{2 - x}$
37. $y = \begin{cases} \sqrt{-x}, & -4 \le x \le 0 \\ \sqrt{x}, & 0 < x \le 4 \end{cases}$
38. $y = \begin{cases} -x - 2, & -2 \le x \le -1 \\ x, & -1 < x \le 1 \\ -x + 2, & 1 < x \le 2 \end{cases}$

In Exercises 39 and 40, write a piecewise formula for the function.

39.


40.


In Exercises 41 and 42, find

(a) $(f \circ g)(-1)$ **(b)** $(g \circ f)(2)$ **(c)** $(f \circ f)(x)$ **(d)** $(g \circ g)(x)$

41. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{\sqrt{x + 2}}$
42. $f(x) = 2 - x, \quad g(x) = \sqrt[3]{x + 1}$

In Exercises 43 and 44, **(a)** write a formula for $f \circ g$ and $g \circ f$ and find the **(b)** domain and **(c)** range of each.

43. $f(x) = 2 - x^2, \quad g(x) = \sqrt{x + 2}$
44. $f(x) = \sqrt{x}, \quad g(x) = \sqrt{1 - x}$

In Exercises 45–48, a parametrization is given for a curve.

(a) Graph the curve. Identify the initial and terminal points, if any. Indicate the direction in which the curve is traced.

(b) Find a Cartesian equation for a curve that contains the parametrized curve. What portion of the graph of the Cartesian equation is traced by the parametrized curve?

45. $x = 5\cos t, \quad y = 2\sin t, \quad 0 \le t \le 2\pi$
46. $x = 4\cos t, \quad y = 4\sin t, \quad \pi/2 \le t < 3\pi/2$
47. $x = 2 - t, \quad y = 11 - 2t, \quad -2 \le t \le 4$
48. $x = 1 + t, \quad y = \sqrt{4 - 2t}, \quad t \le 2$

In Exercises 49–52, give a parametrization for the curve.

49. the line segment with endpoints $(-2, 5)$ and $(4, 3)$
50. the line through $(-3, -2)$ and $(4, -1)$
51. the ray with initial point $(2, 5)$ that passes through $(-1, 0)$
52. $y = x(x - 4), \quad x \le 2$

Group Activity In Exercises 53 and 54, do the following.

(a) Find f^{-1} and show that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.
(b) Graph f and f^{-1} in the same viewing window.

53. $f(x) = 2 - 3x$
54. $f(x) = (x + 2)^2, \quad x \ge -2$

In Exercises 55 and 56, find the measure of the angle in radians and degrees.

55. $\sin^{-1}(0.6)$
56. $\tan^{-1}(-2.3)$
57. Find the six trigonometric values of $\theta = \cos^{-1}(3/7)$. Give exact answers.
58. Solve the equation $\sin x = -0.2$ in the following intervals.
    (a) $0 \le x < 2\pi$ **(b)** $-\infty < x < \infty$
59. Solve for x: $e^{-0.2x} = 4$

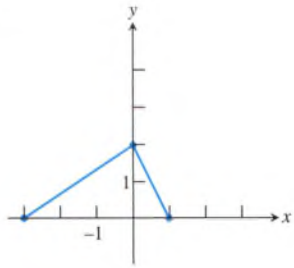
60. The graph of f is shown. Draw the graph of each function.

(a) $y = f(-x)$

(b) $y = -f(x)$

(c) $y = -2f(x + 1) + 1$

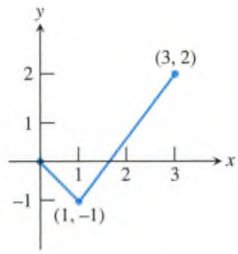
(d) $y = 3f(x - 2) - 2$



61. A portion of the graph of a function defined on $[-3, 3]$ is shown. Complete the graph assuming that the function is

(a) even.

(b) odd.



62. ***Depreciation*** Smith Hauling purchased an 18-wheel truck for \$100,000. The truck depreciates at the constant rate of \$10,000 per year for 10 years.

(a) Write an expression that gives the value y after x years.

(b) When is the value of the truck \$55,000?

63. ***Drug Absorption*** A drug is administered intravenously for pain. The function

$$f(t) = 90 - 52 \ln(1 + t), \quad 0 \le t \le 4$$

gives the number of units of the drug in the body after t hours.

(a) What was the initial number of units of the drug administered?

(b) How much is present after 2 hours? **(c)** Draw the graph of f.

64. ***Finding Time*** If Joenita invests \$1500 in a retirement account that earns 8% compounded annually, how long will it take this single payment to grow to \$5000?

65. ***Guppy Population*** The number of guppies in Susan's aquarium doubles every day. There are four guppies initially.

(a) Write the number of guppies as a function of time t.

(b) How many guppies were present after 4 days? after 1 week?

(c) When will there be 2000 guppies?

(d) **Writing to Learn** Give reasons why this might not be a good model for the growth of Susan's guppy population.

66. ***Doctoral Degrees*** Table 1.23 shows the number of doctoral degrees earned by Hispanic students for several years. Let $x = 0$ represent 1980, $x = 1$ represent 1981, and so forth.

Table 1.23 Doctorates Earned by Hispanic Americans

Year	Number of Degrees
1981	456
1985	677
1990	780
1995	984
2000	1305

Source: Statistical Abstract of the United States, 2004–2005.

(a) Find a linear regression equation for the data and superimpose its graph on a scatter plot of the data.

(b) Use the regression equation to predict the number of doctoral degrees that will be earned by Hispanic Americans in 2002. How close is the estimate to the actual number in 2002 of 1432?

(c) **Writing to Learn** Find the slope of the regression line. What does the slope represent?

67. ***Population of New York*** Table 1.24 shows the population of New York State for several years. Let $x = 0$ represent 1980, $x = 1$ represent 1981, and so forth.

Table 1.24 Population of New York State

Year	Population (thousands)
1980	17,558
1990	17,991
1995	18,524
1998	18,756
1999	18,883
2000	18,977

Source: Statistical Abstract of the United States, 2004–2005.

(a) Find the exponential regression equation for the data and superimpose its graph on a scatter plot of the data.

(b) Use the regression equation to predict the population in 2003. How close is the estimate to the actual number in 2003 of 19,190 thousand?

(c) Use the exponential regression equation to estimate the annual rate of growth of the population of New York State.

AP* Examination Preparation

You may use a graphing calculator to solve the following problems.

68. Consider the point $P(-2, 1)$ and the line L: $x + y = 2$.

(a) Find the slope of L.

(b) Write an equation for the line through P and parallel to L.

(c) Write an equation for the line through P and perpendicular to L.

(d) What is the x-intercept of L?

69. Let $f(x) = 1 - \ln(x - 2)$.

(a) What is the domain of f? **(b)** What is the range of f?

(c) What are the x-intercepts of the graph of f?

(d) Find f^{-1}. **(e)** Confirm your answer algebraically in part (d).

70. Let $f(x) = 1 - 3\cos(2x)$.

(a) What is the domain of f? **(b)** What is the range of f?

(c) What is the period of f?

(d) Is f an even function, odd function, or neither?

(e) Find all the zeros of f in $\pi/2 \le x \le \pi$.

Chapter 2

Limits and Continuity

An Economic Injury Level (EIL) is a measurement of the fewest number of insect pests that will cause economic damage to a crop or forest. It has been estimated that monitoring pest populations and establishing EILs can reduce pesticide use by 30%–50%.

Accurate population estimates are crucial for determining EILs. A population density of one insect pest can be approximated by

$$D(t) = \frac{t^2}{90} + \frac{t}{3}$$

pests per plant, where t is the number of days since initial infestation. What is the rate of change of this population density when the population density is equal to the EIL of 20 pests per plant? Section 2.4 can help answer this question.

Chapter 2 Overview

The concept of limit is one of the ideas that distinguish calculus from algebra and trigonometry.

In this chapter, we show how to define and calculate limits of function values. The calculation rules are straightforward and most of the limits we need can be found by substitution, graphical investigation, numerical approximation, algebra, or some combination of these.

One of the uses of limits is to test functions for continuity. Continuous functions arise frequently in scientific work because they model such an enormous range of natural behavior. They also have special mathematical properties, not otherwise guaranteed.

2.1 Rates of Change and Limits

What you'll learn about

- Average and Instantaneous Speed
- Definition of Limit
- Properties of Limits
- One-sided and Two-sided Limits
- Sandwich Theorem

. . . and why

Limits can be used to describe continuity, the derivative, and the integral: the ideas giving the foundation of calculus.

Average and Instantaneous Speed

A moving body's **average speed** during an interval of time is found by dividing the distance covered by the elapsed time. The unit of measure is length per unit time—kilometers per hour, feet per second, or whatever is appropriate to the problem at hand.

EXAMPLE 1 Finding an Average Speed

A rock breaks loose from the top of a tall cliff. What is its average speed during the first 2 seconds of fall?

SOLUTION

Experiments show that a dense solid object dropped from rest to fall freely near the surface of the earth will fall

$$y = 16t^2$$

feet in the first t seconds. The average speed of the rock over any given time interval is the distance traveled, Δy, divided by the length of the interval Δt. For the first 2 seconds of fall, from $t = 0$ to $t = 2$, we have

$$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(0)^2}{2 - 0} = 32\frac{\text{ft}}{\text{sec}}.$$

Now try Exercise 1.

Free Fall

Near the surface of the earth, all bodies fall with the same constant acceleration. The distance a body falls after it is released from rest is a constant multiple of the square of the time fallen. At least, that is what happens when a body falls in a vacuum, where there is no air to slow it down. The square-of-time rule also holds for dense, heavy objects like rocks, ball bearings, and steel tools during the first few seconds of fall through air, before the velocity builds up to where air resistance begins to matter. When air resistance is absent or insignificant and the only force acting on a falling body is the force of gravity, we call the way the body falls *free fall*.

EXAMPLE 2 Finding an Instantaneous Speed

Find the speed of the rock in Example 1 at the instant $t = 2$.

SOLUTION

Solve Numerically We can calculate the average speed of the rock over the interval from time $t = 2$ to any slightly later time $t = 2 + h$ as

$$\frac{\Delta y}{\Delta t} = \frac{16(2 + h)^2 - 16(2)^2}{h}. \tag{1}$$

We cannot use this formula to calculate the speed at the exact instant $t = 2$ because that would require taking $h = 0$, and $0/0$ is undefined. However, we can get a good idea of what is happening at $t = 2$ by evaluating the formula at values of h *close* to 0. When we do, we see a clear pattern (Table 2.1 on the next page). As h approaches 0, the average speed approaches the limiting value 64 ft/sec.

continued

Table 2.1 Average Speeds over Short Time Intervals Starting at $t = 2$

$$\frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h}$$

Length of Time Interval, h (sec)	Average Speed for Interval $\Delta y/\Delta t$ (ft/sec)
1	80
0.1	65.6
0.01	64.16
0.001	64.016
0.0001	64.0016
0.00001	64.00016

Confirm Algebraically If we expand the numerator of Equation 1 and simplify, we find that

$$\frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h} = \frac{16(4 + 4h + h^2) - 64}{h}$$

$$= \frac{64h + 16h^2}{h} = 64 + 16h.$$

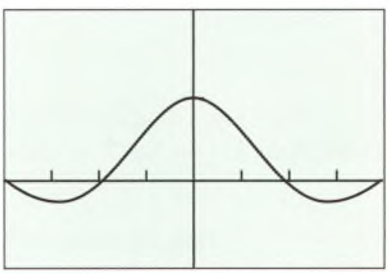
For values of h different from 0, the expressions on the right and left are equivalent and the average speed is $64 + 16h$ ft/sec. We can now see why the average speed has the limiting value $64 + 16(0) = 64$ ft/sec as h approaches 0. *Now try Exercise 3.*

Definition of Limit

As in the preceding example, most limits of interest in the real world can be viewed as numerical limits of values of functions. And this is where a graphing utility and calculus come in. A calculator can suggest the limits, and calculus can give the mathematics for confirming the limits analytically.

Limits give us a language for describing how the outputs of a function behave as the inputs approach some particular value. In Example 2, the average speed was not defined at $h = 0$ but approached the limit 64 as h approached 0. We were able to see this numerically and to confirm it algebraically by eliminating h from the denominator. But we cannot always do that. For instance, we can see both graphically and numerically (Figure 2.1) that the values of $f(x) = (\sin x)/x$ approach 1 as x approaches 0.

We cannot eliminate the x from the denominator of $(\sin x)/x$ to confirm the observation algebraically. We need to use a theorem about limits to make that confirmation, as you will see in Exercise 75.



$[-2\pi, 2\pi]$ by $[-1, 2]$

(a)

X	Y1	
-.3	.98507	
-.2	.99335	
-.1	.99833	
0	ERROR	
.1	.99833	
.2	.99335	
.3	.98507	

Y1 = sin(X)/X

(b)

Figure 2.1 (a) A graph and (b) table of values for $f(x) = (\sin x)/x$ that suggest the limit of f as x approaches 0 is 1.

DEFINITION Limit

Assume f is defined in a neighborhood of c and let c and L be real numbers. The function ***f* has limit *L* as *x* approaches *c*** if, given any positive number ε, there is a positive number δ such that for all x,

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

We write

$$\lim_{x \to c} f(x) = L.$$

The sentence $\lim_{x\to c} f(x) = L$ is read, "The limit of f of x as x approaches c equals L." The notation means that the values $f(x)$ of the function f approach or equal L as the values of x approach *(but do not equal)* c. Appendix A3 provides practice applying the definition of limit.

We saw in Example 2 that $\lim_{h\to 0} (64 + 16h) = 64$.

As suggested in Figure 2.1,

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Figure 2.2 illustrates the fact that the existence of a limit as $x \to c$ never depends on how the function may or may not be defined at c. The function f has limit 2 as $x \to 1$ even though f is not defined at 1. The function g has limit 2 as $x \to 1$ even though $g(1) \neq 2$. The function h is the only one whose limit as $x \to 1$ equals its value at $x = 1$.

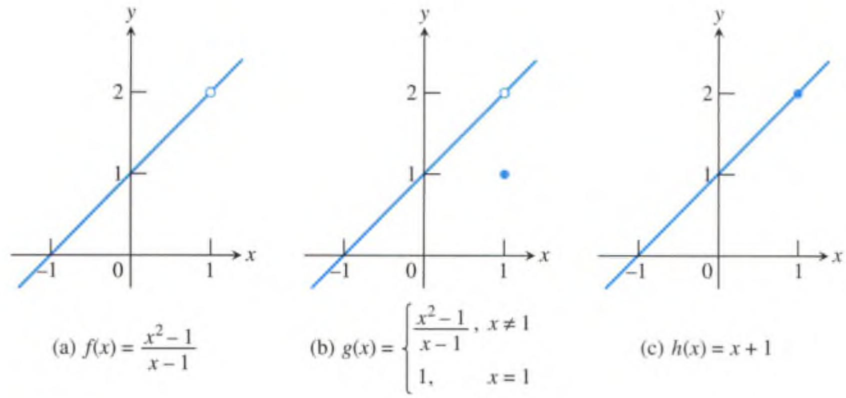


Figure 2.2 $\lim_{x \to 1} f(x) = \lim_{x \to 1} g(x) = \lim_{x \to 1} h(x) = 2$

Properties of Limits

By applying six basic facts about limits, we can calculate many unfamiliar limits from limits we already know. For instance, from knowing that

$$\lim_{x \to c} (k) = k \qquad \text{Limit of the function with constant value } k$$

and

$$\lim_{x \to c} (x) = c, \qquad \text{Limit of the identity function at } x = c$$

we can calculate the limits of all polynomial and rational functions. The facts are listed in Theorem 1.

THEOREM 1 Properties of Limits

If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \text{ then}$$

1. *Sum Rule:* $\lim_{x \to c} (f(x) + g(x)) = L + M$

The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule:* $\lim_{x \to c} (f(x) - g(x)) = L - M$

The limit of the difference of two functions is the difference of their limits.

3. *Product Rule:* $\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M$

The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:* $\lim_{x \to c} (k \cdot f(x)) = k \cdot L$

The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule:* $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \; M \neq 0$

The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

continued

6. *Power Rule:* If r and s are integers, $s \neq 0$, then

$$\lim_{x \to c} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number.

The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

Here are some examples of how Theorem 1 can be used to find limits of polynomial and rational functions.

EXAMPLE 3 Using Properties of Limits

Use the observations $\lim_{x \to c} k = k$ and $\lim_{x \to c} x = c$, and the properties of limits to find the following limits.

(a) $\lim_{x \to c} (x^3 + 4x^2 - 3)$ **(b)** $\lim_{x \to c} \dfrac{x^4 + x^2 - 1}{x^2 + 5}$

SOLUTION

(a) $$\lim_{x \to c} (x^3 + 4x^2 - 3) = \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3$$ Sum and Difference Rules

$$= c^3 + 4c^2 - 3$$ Product and Constant Multiple Rules

(b) $$\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \to c} (x^4 + x^2 - 1)}{\lim_{x \to c} (x^2 + 5)}$$ Quotient Rule

$$= \frac{\lim_{x \to c} x^4 + \lim_{x \to c} x^2 - \lim_{x \to c} 1}{\lim_{x \to c} x^2 + \lim_{x \to c} 5}$$ Sum and Difference Rules

$$= \frac{c^4 + c^2 - 1}{c^2 + 5}$$ Product Rule

Now try Exercises 5 and 6.

Example 3 shows the remarkable strength of Theorem 1. From the two simple observations that $\lim_{x \to c} k = k$ and $\lim_{x \to c} x = c$, we can immediately work our way to limits of polynomial functions and most rational functions using substitution.

THEOREM 2 Polynomial and Rational Functions

1. If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ is any polynomial function and c is any real number, then

$$\lim_{x \to c} f(x) = f(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

2. If $f(x)$ and $g(x)$ are polynomials and c is any real number, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}, \quad \text{provided that } g(c) \neq 0.$$

EXAMPLE 4 Using Theorem 2

(a) $\lim_{x \to 3} [x^2(2 - x)] = (3)^2(2 - 3) = -9$

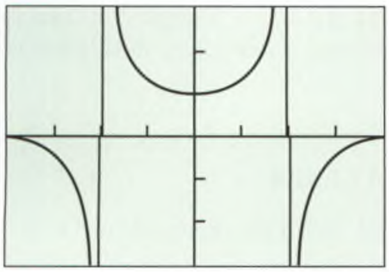
(b) $\lim_{x \to 2} \dfrac{x^2 + 2x + 4}{x + 2} = \dfrac{(2)^2 + 2(2) + 4}{2 + 2} = \dfrac{12}{4} = 3$

Now try Exercises 9 and 11.

As with polynomials, limits of many familiar functions can be found by substitution at points where they are defined. This includes trigonometric functions, exponential and logarithmic functions, and composites of these functions. Feel free to use these properties.

EXAMPLE 5 Using the Product Rule

Determine $\lim_{x \to 0} \dfrac{\tan x}{x}$.



$[-\pi, \pi]$ by $[-3, 3]$

Figure 2.3 The graph of $f(x) = (\tan x)/x$ suggests that $f(x) \to 1$ as $x \to 0$. (Example 5)

SOLUTION

Solve Graphically The graph of $f(x) = (\tan x)/x$ in Figure 2.3 suggests that the limit exists and is about 1.

Confirm Analytically Using the analytic result of Exercise 75, we have

$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \left(\frac{\sin x}{x} \cdot \frac{1}{\cos x} \right) \qquad \tan x = \frac{\sin x}{\cos x}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{\cos x} \qquad \text{Product Rule}$$

$$= 1 \cdot \frac{1}{\cos 0} = 1 \cdot \frac{1}{1} = 1.$$

Now try Exercise 27.

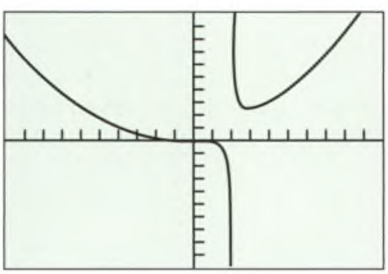
Sometimes we can use a graph to discover that limits do not exist, as illustrated by Example 6.

EXAMPLE 6 Exploring a Nonexistent Limit

Use a graph to show that

$$\lim_{x \to 2} \frac{x^3 - 1}{x - 2}$$

does not exist.



$[-10, 10]$ by $[-100, 100]$

Figure 2.4 The graph of $f(x) = (x^3 - 1)/(x - 2)$ obtained using parametric graphing to produce a more accurate graph. (Example 6)

SOLUTION

Notice that the denominator is 0 when x is replaced by 2, so we cannot use substitution to determine the limit. The graph in Figure 2.4 of $f(x) = (x^3 - 1)/(x - 2)$ strongly suggests that as $x \to 2$ from either side, the absolute values of the function values get very large. This, in turn, suggests that the limit does not exist.

Now try Exercise 29.

One-sided and Two-sided Limits

Sometimes the values of a function f tend to different limits as x approaches a number c from opposite sides. When this happens, we call the limit of f as x approaches c from the right the **right-hand limit** of f at c and the limit as x approaches c from the left the **left-hand limit** of f at c. Here is the notation we use:

right-hand: $\lim_{x \to c^+} f(x)$ *The limit of f as x approaches c from the right.*

left-hand: $\lim_{x \to c^-} f(x)$ *The limit of f as x approaches c from the left.*

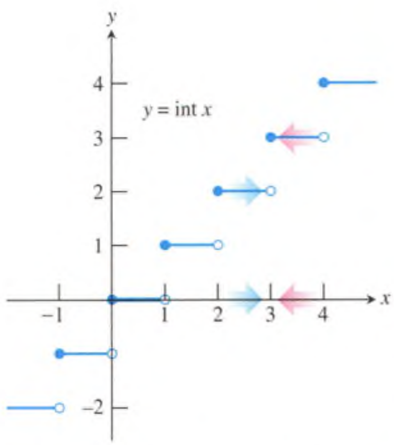


Figure 2.5 At each integer, the greatest integer function $y = \text{int } x$ has different right-hand and left-hand limits. (Example 7)

EXAMPLE 7 Function Values Approach Two Numbers

The greatest integer function $f(x) = \text{int } x$ has different right-hand and left-hand limits at each integer, as we can see in Figure 2.5. For example,

$$\lim_{x \to 3^+} \text{int } x = 3 \quad \text{and} \quad \lim_{x \to 3^-} \text{int } x = 2.$$

The limit of int x as x approaches an integer n from the right is n, while the limit as x approaches n from the left is $n - 1$.

Now try Exercises 31 and 32.

We sometimes call $\lim_{x \to c} f(x)$ the **two-sided limit** of f at c to distinguish it from the *one-sided* right-hand and left-hand limits of f at c. Theorem 3 shows how these limits are related.

On the Far Side

If f is not defined to the left of $x = c$, then f does not have a left-hand limit at c. Similarly, if f is not defined to the right of $x = c$, then f does not have a right-hand limit at c.

THEOREM 3 One-sided and Two-sided Limits

A function $f(x)$ has a limit as x approaches c if and only if the right-hand and left-hand limits at c exist and are equal. In symbols,

$$\lim_{x \to c} f(x) = L \Leftrightarrow \lim_{x \to c^+} f(x) = L \quad \text{and} \quad \lim_{x \to c^-} f(x) = L.$$

Thus, the greatest integer function $f(x) = \text{int } x$ of Example 7 does not have a limit as $x \to 3$ even though each one-sided limit exists.

EXAMPLE 8 Exploring Right- and Left-Hand Limits

All the following statements about the function $y = f(x)$ graphed in Figure 2.6 are true.

At $x = 0$: $\lim_{x \to 0^+} f(x) = 1$.

At $x = 1$: $\lim_{x \to 1^-} f(x) = 0$ even though $f(1) = 1$,

$\lim_{x \to 1^+} f(x) = 1$,

f has no limit as $x \to 1$. (The right- and left-hand limits at 1 are not equal, so $\lim_{x \to 1} f(x)$ does not exist.)

At $x = 2$: $\lim_{x \to 2^-} f(x) = 1$,

$\lim_{x \to 2^+} f(x) = 1$,

$\lim_{x \to 2} f(x) = 1$ even though $f(2) = 2$.

At $x = 3$: $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = 2 = f(3) = \lim_{x \to 3} f(x)$.

At $x = 4$: $\lim_{x \to 4^-} f(x) = 1$.

At noninteger values of c between 0 and 4, f has a limit as $x \to c$.

Now try Exercise 37.

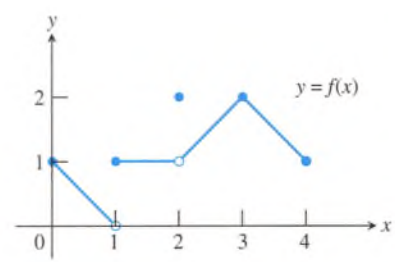


Figure 2.6 The graph of the function

$$f(x) = \begin{cases} -x + 1, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \\ x - 1, & 2 < x \le 3 \\ -x + 5, & 3 < x \le 4. \end{cases}$$

(Example 8)

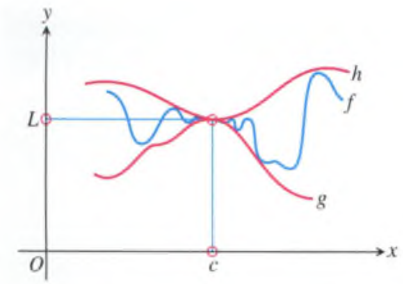


Figure 2.7 Sandwiching f between g and h forces the limiting value of f to be between the limiting values of g and h.

Sandwich Theorem

If we cannot find a limit directly, we may be able to find it indirectly with the Sandwich Theorem. The theorem refers to a function f whose values are sandwiched between the values of two other functions, g and h. If g and h have the same limit as $x \to c$, then f has that limit too, as suggested by Figure 2.7.

THEOREM 4 The Sandwich Theorem

If $g(x) \le f(x) \le h(x)$ for all $x \ne c$ in some interval about c, and

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L,$$

then

$$\lim_{x \to c} f(x) = L.$$

EXAMPLE 9 Using the Sandwich Theorem

Show that $\lim_{x \to 0} [x^2 \sin (1/x)] = 0$.

SOLUTION

We know that the values of the sine function lie between –1 and 1. So, it follows that

$$\left| x^2 \sin \frac{1}{x} \right| = |x^2| \cdot \left| \sin \frac{1}{x} \right| \le |x^2| \cdot 1 = x^2$$

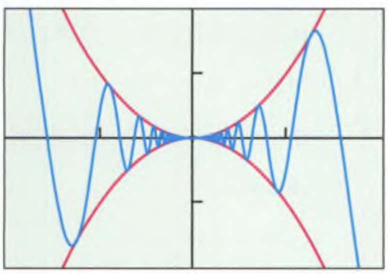
and

$$-x^2 \le x^2 \sin \frac{1}{x} \le x^2.$$

Because $\lim_{x \to 0} (-x^2) = \lim_{x \to 0} x^2 = 0$, the Sandwich Theorem gives

$$\lim_{x \to 0} \left(x^2 \sin \frac{1}{x} \right) = 0.$$

The graphs in Figure 2.8 support this result.



[–0.2, 0.2] by [–0.02, 0.02]

Figure 2.8 The graphs of $y_1 = x^2$, $y_2 = x^2 \sin (1/x)$, and $y_3 = -x^2$. Notice that $y_3 \le y_2 \le y_1$. (Example 9)

Quick Review 2.1 *(For help, go to Section 1.2.)*

In Exercises 1–4, find $f(2)$.

1. $f(x) = 2x^3 - 5x^2 + 4$

2. $f(x) = \dfrac{4x^2 - 5}{x^3 + 4}$

3. $f(x) = \sin\left(\pi \dfrac{x}{2}\right)$

4. $f(x) = \begin{cases} 3x - 1, & x < 2 \\ \dfrac{1}{x^2 - 1}, & x \ge 2 \end{cases}$

In Exercises 5–8, write the inequality in the form $a < x < b$.

5. $|x| < 4$

6. $|x| < c^2$

7. $|x - 2| < 3$

8. $|x - c| < d^2$

In Exercises 9 and 10, write the fraction in reduced form.

9. $\dfrac{x^2 - 3x - 18}{x + 3}$

10. $\dfrac{2x^2 - x}{2x^2 + x - 1}$

Section 2.1 Exercises

In Exercises 1–4, an object dropped from rest from the top of a tall building falls $y = 16t^2$ feet in the first t seconds.

1. Find the average speed during the first 3 seconds of fall.
2. Find the average speed during the first 4 seconds of fall.
3. Find the speed of the object at $t = 3$ seconds and confirm your answer algebraically.
4. Find the speed of the object at $t = 4$ seconds and confirm your answer algebraically.

In Exercises 5 and 6, use $\lim_{x\to c} k = k$, $\lim_{x\to c} x = c$, and the properties of limits to find the limit.

5. $\lim_{x\to c} (2x^3 - 3x^2 + x - 1)$

6. $\lim_{x\to c} \dfrac{x^4 - x^3 + 1}{x^2 + 9}$

In Exercises 7–14, determine the limit by substitution. Support graphically.

7. $\lim_{x\to -1/2} 3x^2(2x - 1)$

8. $\lim_{x\to -4} (x + 3)^{1998}$

9. $\lim_{x\to 1} (x^3 + 3x^2 - 2x - 17)$

10. $\lim_{y\to 2} \dfrac{y^2 + 5y + 6}{y + 2}$

11. $\lim_{y\to -3} \dfrac{y^2 + 4y + 3}{y^2 - 3}$

12. $\lim_{x\to 1/2} \text{int}\, x$

13. $\lim_{x\to -2} (x - 6)^{2/3}$

14. $\lim_{x\to 2} \sqrt{x + 3}$

In Exercises 15–18, explain why you cannot use substitution to determine the limit. Find the limit if it exists.

15. $\lim_{x\to -2} \sqrt{x - 2}$

16. $\lim_{x\to 0} \dfrac{1}{x^2}$

17. $\lim_{x\to 0} \dfrac{|x|}{x}$

18. $\lim_{x\to 0} \dfrac{(4 + x)^2 - 16}{x}$

In Exercises 19–28, determine the limit graphically. Confirm algebraically.

19. $\lim_{x\to 1} \dfrac{x - 1}{x^2 - 1}$

20. $\lim_{t\to 2} \dfrac{t^2 - 3t + 2}{t^2 - 4}$

21. $\lim_{x\to 0} \dfrac{5x^3 + 8x^2}{3x^4 - 16x^2}$

22. $\lim_{x\to 0} \dfrac{\dfrac{1}{2 + x} - \dfrac{1}{2}}{x}$

23. $\lim_{x\to 0} \dfrac{(2 + x)^3 - 8}{x}$

24. $\lim_{x\to 0} \dfrac{\sin 2x}{x}$

25. $\lim_{x\to 0} \dfrac{\sin x}{2x^2 - x}$

26. $\lim_{x\to 0} \dfrac{x + \sin x}{x}$

27. $\lim_{x\to 0} \dfrac{\sin^2 x}{x}$

28. $\lim_{x\to 0} \dfrac{3 \sin 4x}{\sin 3x}$

In Exercises 29 and 30, use a graph to show that the limit does not exist.

29. $\lim_{x\to 1} \dfrac{x^2 - 4}{x - 1}$

30. $\lim_{x\to 2} \dfrac{x + 1}{x^2 - 4}$

In Exercises 31–36, determine the limit.

31. $\lim_{x\to 0^+} \text{int}\, x$

32. $\lim_{x\to 0^-} \text{int}\, x$

33. $\lim_{x\to 0.01} \text{int}\, x$

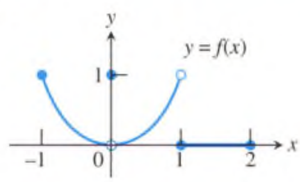
34. $\lim_{x\to 2^-} \text{int}\, x$

35. $\lim_{x\to 0^+} \dfrac{x}{|x|}$

36. $\lim_{x\to 0^-} \dfrac{x}{|x|}$

In Exercises 37 and 38, which of the statements are true about the function $y = f(x)$ graphed there, and which are false?

37.



(a) $\lim_{x\to -1^+} f(x) = 1$

(b) $\lim_{x\to 0^-} f(x) = 0$

(c) $\lim_{x\to 0^-} f(x) = 1$

(d) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

(e) $\lim_{x\to 0} f(x)$ exists

(f) $\lim_{x\to 0} f(x) = 0$

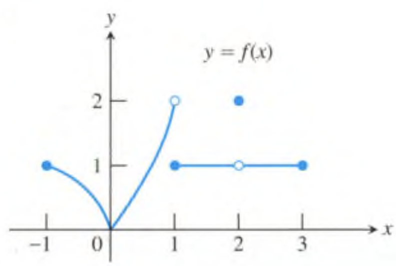
(g) $\lim_{x\to 0} f(x) = 1$

(h) $\lim_{x\to 1} f(x) = 1$

(i) $\lim_{x\to 1} f(x) = 0$

(j) $\lim_{x\to 2^-} f(x) = 2$

38.



(a) $\lim_{x\to -1^+} f(x) = 1$

(b) $\lim_{x\to 2} f(x)$ does not exist.

(c) $\lim_{x\to 2} f(x) = 2$

(d) $\lim_{x\to 1^-} f(x) = 2$

(e) $\lim_{x\to 1^+} f(x) = 1$

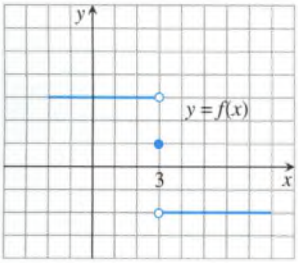
(f) $\lim_{x\to 1} f(x)$ does not exist.

(g) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$

(h) $\lim_{x\to c} f(x)$ exists at every c in $(-1, 1)$.

(i) $\lim_{x\to c} f(x)$ exists at every c in $(1, 3)$.

In Exercises 39–44, use the graph to estimate the limits and value of the function, or explain why the limits do not exist.

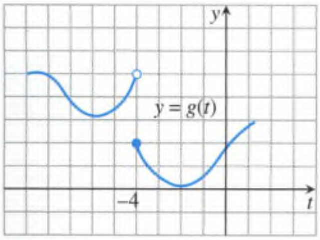
39. 

(a) $\lim_{x\to 3^-} f(x)$

(b) $\lim_{x\to 3^+} f(x)$

(c) $\lim_{x\to 3} f(x)$

(d) $f(3)$

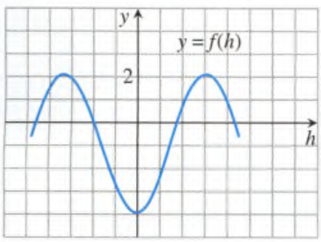
40. 

(a) $\lim_{t\to -4^-} g(t)$

(b) $\lim_{t\to -4^+} g(t)$

(c) $\lim_{t\to -4} g(t)$

(d) $g(-4)$

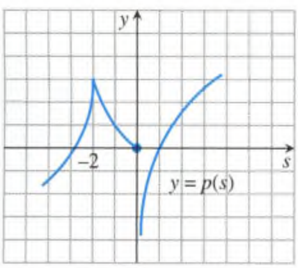
41. 

(a) $\lim_{h\to 0^-} f(h)$

(b) $\lim_{h\to 0^+} f(h)$

(c) $\lim_{h\to 0} f(h)$

(d) $f(0)$

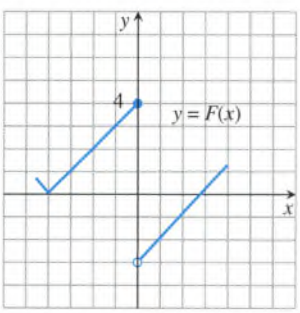
42. 

(a) $\lim_{s\to -2^-} p(s)$

(b) $\lim_{s\to -2^+} p(s)$

(c) $\lim_{s\to -2} p(s)$

(d) $p(-2)$

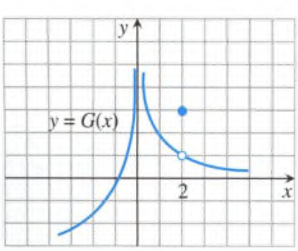
43. 

(a) $\lim_{x\to 0^-} F(x)$

(b) $\lim_{x\to 0^+} F(x)$

(c) $\lim_{x\to 0} F(x)$

(d) $F(0)$

44. 

(a) $\lim_{x\to 2^-} G(x)$

(b) $\lim_{x\to 2^+} G(x)$

(c) $\lim_{x\to 2} G(x)$

(d) $G(2)$

In Exercises 45–48, match the function with the table.

45. $y_1 = \dfrac{x^2 + x - 2}{x - 1}$ **46.** $y_1 = \dfrac{x^2 - x - 2}{x - 1}$

47. $y_1 = \dfrac{x^2 - 2x + 1}{x - 1}$ **48.** $y_1 = \dfrac{x^2 + x - 2}{x + 1}$

X	Y1	
.7	−.4765	
.8	−.3111	
.9	−.1526	
1	0	
1.1	.14762	
1.2	.29091	
1.3	.43043	
X = .7		

(a)

X	Y1	
.7	7.3667	
.8	10.8	
.9	20.9	
1	ERROR	
1.1	−18.9	
1.2	−8.8	
1.3	−5.367	
X = .7		

(b)

X	Y1	
.7	2.7	
.8	2.8	
.9	2.9	
1	ERROR	
1.1	3.1	
1.2	3.2	
1.3	3.3	
X = .7		

(c)

X	Y1	
.7	−.3	
.8	−.2	
.9	−.1	
1	ERROR	
1.1	.1	
1.2	.2	
1.3	.3	
X = .7		

(d)

In Exercises 49 and 50, determine the limit.

49. Assume that $\lim_{x\to 4} f(x) = 0$ and $\lim_{x\to 4} g(x) = 3$.

(a) $\lim_{x\to 4} (g(x) + 3)$ **(b)** $\lim_{x\to 4} x f(x)$

(c) $\lim_{x\to 4} g^2(x)$ **(d)** $\lim_{x\to 4} \dfrac{g(x)}{f(x) - 1}$

50. Assume that $\lim_{x\to b} f(x) = 7$ and $\lim_{x\to b} g(x) = -3$.

(a) $\lim_{x\to b} (f(x) + g(x))$ **(b)** $\lim_{x\to b} (f(x) \cdot g(x))$

(c) $\lim_{x\to b} 4\, g(x)$ **(d)** $\lim_{x\to b} \dfrac{f(x)}{g(x)}$

In Exercises 51–54, complete parts (a), (b), and (c) for the piecewise-defined function.

(a) Draw the graph of f.

(b) Determine $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$.

(c) Writing to Learn Does $\lim_{x\to c} f(x)$ exist? If so, what is it? If not, explain.

51. $c = 2, f(x) = \begin{cases} 3 - x, & x < 2 \\ \dfrac{x}{2} + 1, & x > 2 \end{cases}$

52. $c = 2, f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ x/2, & x > 2 \end{cases}$

53. $c = 1, f(x) = \begin{cases} \dfrac{1}{x - 1}, & x < 1 \\ x^3 - 2x + 5, & x \ge 1 \end{cases}$

54. $c = -1, f(x) = \begin{cases} 1 - x^2, & x \ne -1 \\ 2, & x = -1 \end{cases}$

In Exercises 55–58, complete parts (a)–(d) for the piecewise-defined function.

(a) Draw the graph of f.

(b) At what points c in the domain of f does $\lim_{x\to c} f(x)$ exist?

(c) At what points c does only the left-hand limit exist?

(d) At what points c does only the right-hand limit exist?

55. $f(x) = \begin{cases} \sin x, & -2\pi \le x < 0 \\ \cos x, & 0 \le x \le 2\pi \end{cases}$

56. $f(x) = \begin{cases} \cos x, & -\pi \le x < 0 \\ \sec x, & 0 \le x \le \pi \end{cases}$

57. $f(x) = \begin{cases} \sqrt{1-x^2}, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \end{cases}$

58. $f(x) = \begin{cases} x, & -1 \le x < 0, \text{ or } 0 < x \le 1 \\ 1, & x = 0 \\ 0, & x < -1, \text{ or } x > 1 \end{cases}$

In Exercises 59–62, find the limit graphically. Use the Sandwich Theorem to confirm your answer.

59. $\lim_{x\to 0} x \sin x$

60. $\lim_{x\to 0} x^2 \sin x$

61. $\lim_{x\to 0} x^2 \sin \dfrac{1}{x^2}$

62. $\lim_{x\to 0} x^2 \cos \dfrac{1}{x^2}$

63. ***Free Fall*** A water balloon dropped from a window high above the ground falls $y = 4.9t^2$ m in t sec. Find the balloon's

(a) average speed during the first 3 sec of fall.

(b) speed at the instant $t = 3$.

64. ***Free Fall on a Small Airless Planet*** A rock released from rest to fall on a small airless planet falls $y = gt^2$ m in t sec, g a constant. Suppose that the rock falls to the bottom of a crevasse 20 m below and reaches the bottom in 4 sec.

(a) Find the value of g.

(b) Find the average speed for the fall.

(c) With what speed did the rock hit the bottom?

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

65. True or False If $\lim_{x\to c^-} f(x) = 2$ and $\lim_{x\to c^+} f(x) = 2$, then $\lim_{x\to c} f(x) = 2$. Justify your answer.

66. True or False $\lim_{x\to 0} \dfrac{x + \sin x}{x} = 2$. Justify your answer.

In Exercises 67–70, use the following function.

$$f(x) = \begin{cases} 2 - x, & x \le 1 \\ \dfrac{x}{2} + 1, & x > 1 \end{cases}$$

67. Multiple Choice What is the value of $\lim_{x\to 1^-} f(x)$?

(A) 5/2 **(B)** 3/2 **(C)** 1 **(D)** 0 **(E)** does not exist

68. Multiple Choice What is the value of $\lim_{x\to 1^+} f(x)$?

(A) 5/2 **(B)** 3/2 **(C)** 1 **(D)** 0 **(E)** does not exist

69. Multiple Choice What is the value of $\lim_{x\to 1} f(x)$?

(A) 5/2 **(B)** 3/2 **(C)** 1 **(D)** 0 **(E)** does not exist

70. Multiple Choice What is the value of $f(1)$?

(A) 5/2 **(B)** 3/2 **(C)** 1 **(D)** 0 **(E)** does not exist

Explorations

In Exercises 71–74, complete the following tables and state what you believe $\lim_{x\to 0} f(x)$ to be.

(a)

x	−0.1	−0.01	−0.001	−0.0001	...
$f(x)$	?	?	?	?	

(b)

x	0.1	0.01	0.001	0.0001	...
$f(x)$	?	?	?	?	

71. $f(x) = x \sin \dfrac{1}{x}$

72. $f(x) = \sin \dfrac{1}{x}$

73. $f(x) = \dfrac{10^x - 1}{x}$

74. $f(x) = x \sin(\ln |x|)$

75. Group Activity To prove that $\lim_{\theta\to 0} (\sin \theta)/\theta = 1$ when θ is measured in radians, the plan is to show that the right- and left-hand limits are both 1.

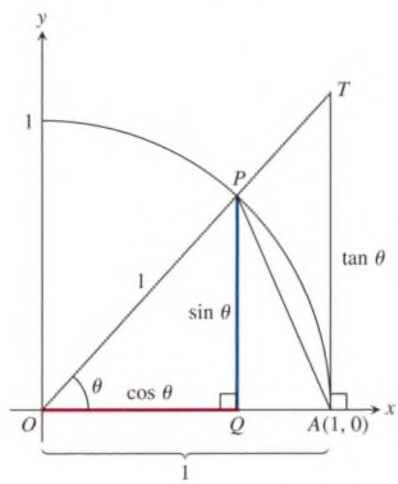
(a) To show that the right-hand limit is 1, explain why we can restrict our attention to $0 < \theta < \pi/2$.

(b) Use the figure to show that

$$\text{area of } \triangle OAP = \frac{1}{2}\sin\theta,$$

$$\text{area of sector } OAP = \frac{\theta}{2},$$

$$\text{area of } \triangle OAT = \frac{1}{2}\tan\theta.$$



(c) Use part (b) and the figure to show that for $0 < \theta < \pi/2$,

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$

(d) Show that for $0 < \theta < \pi/2$ the inequality of part (c) can be written in the form

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

(e) Show that for $0 < \theta < \pi/2$ the inequality of part (d) can be written in the form

$$\cos\theta < \frac{\sin\theta}{\theta} < 1.$$

(f) Use the Sandwich Theorem to show that

$$\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1.$$

(g) Show that $(\sin\theta)/\theta$ is an even function.

(h) Use part (g) to show that

$$\lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = 1.$$

(i) Finally, show that

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$

Extending the Ideas

76. ***Controlling Outputs*** Let $f(x) = \sqrt{3x - 2}$.

(a) Show that $\lim_{x\to 2} f(x) = 2 = f(2)$.

(b) Use a graph to estimate values for a and b so that $1.8 < f(x) < 2.2$ provided $a < x < b$.

(c) Use a graph to estimate values for a and b so that $1.99 < f(x) < 2.01$ provided $a < x < b$.

77. ***Controlling Outputs*** Let $f(x) = \sin x$.

(a) Find $f(\pi/6)$.

(b) Use a graph to estimate an interval (a, b) about $x = \pi/6$ so that $0.3 < f(x) < 0.7$ provided $a < x < b$.

(c) Use a graph to estimate an interval (a, b) about $x = \pi/6$ so that $0.49 < f(x) < 0.51$ provided $a < x < b$.

78. ***Limits and Geometry*** Let $P(a, a^2)$ be a point on the parabola $y = x^2$, $a > 0$. Let O be the origin and $(0, b)$ the y-intercept of the perpendicular bisector of line segment OP. Find $\lim_{P\to O} b$.

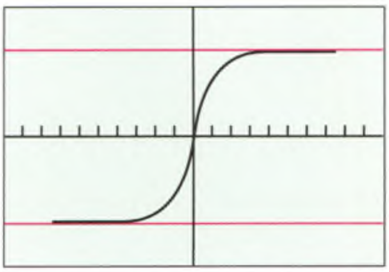
2.2 Limits Involving Infinity

What you'll learn about

- Finite Limits as $x \to \pm\infty$
- Sandwich Theorem Revisited
- Infinite Limits as $x \to a$
- End Behavior Models
- "Seeing" Limits as $x \to \pm\infty$

. . . and why

Limits can be used to describe the behavior of functions for numbers large in absolute value.



[−10, 10] by [−1.5, 1.5]

(a)

X	Y1
0	0
1	.7071
2	.8944
3	.9487
4	.9701
5	.9806
6	.9864
Y1 = X/√(X² + 1)	

X	Y1
−6	−.9864
−5	−.9806
−4	−.9701
−3	−.9487
−2	−.8944
−1	−.7071
0	0
Y1 = X/√(X² + 1)	

(b)

Figure 2.10 (a) The graph of $f(x) = x/\sqrt{x^2+1}$ has two horizontal asymptotes, $y = -1$ and $y = 1$. (b) Selected values of f. (Example 1)

Finite Limits as $x \to \pm\infty$

The symbol for infinity (∞) does not represent a real number. We use ∞ to describe the behavior of a function when the values in its domain or range outgrow all finite bounds. For example, when we say "the limit of f as x approaches infinity" we mean the limit of f as x moves increasingly far to the right on the number line. When we say "the limit of f as x approaches negative infinity ($-\infty$)" we mean the limit of f as x moves increasingly far to the left. (The limit in each case may or may not exist.)

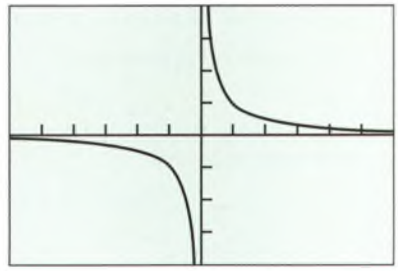
Looking at $f(x) = 1/x$ (Figure 2.9), we observe

(a) as $x \to \infty$, $(1/x) \to 0$ and we write

$$\lim_{x\to\infty} (1/x) = 0,$$

(b) as $x \to -\infty$, $(1/x) \to 0$ and we write

$$\lim_{x\to-\infty} (1/x) = 0.$$



[−6, 6] by [−4, 4]

Figure 2.9 The graph of $f(x) = 1/x$

We say that the line $y = 0$ is a *horizontal asymptote* of the graph of f.

> **DEFINITION Horizontal Asymptote**
>
> The line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either
>
> $$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b.$$

The graph of $f(x) = 2 + (1/x)$ has the single horizontal asymptote $y = 2$ because

$$\lim_{x\to\infty}\left(2 + \frac{1}{x}\right) = 2 \quad \text{and} \quad \lim_{x\to-\infty}\left(2 + \frac{1}{x}\right) = 2.$$

A function can have more than one horizontal asymptote, as Example 1 demonstrates.

EXAMPLE 1 Looking for Horizontal Asymptotes

Use graphs and tables to find $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$, and identify all horizontal asymptotes of $f(x) = x/\sqrt{x^2+1}$.

SOLUTION

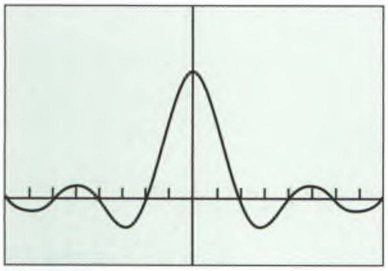
Solve Graphically Figure 2.10a shows the graph for $-10 \le x \le 10$. The graph climbs rapidly toward the line $y = 1$ as x moves away from the origin to the right. On our calculator screen, the graph soon becomes indistinguishable from the line. Thus $\lim_{x\to\infty} f(x) = 1$. Similarly, as x moves away from the origin to the left, the graph drops rapidly toward the line $y = -1$ and soon appears to overlap the line. Thus $\lim_{x\to-\infty} f(x) = -1$. The horizontal asymptotes are $y = 1$ and $y = -1$.

continued

Confirm Numerically The table in Figure 2.10b confirms the rapid approach of $f(x)$ toward 1 as $x \to \infty$. Since f is an odd function of x, we can expect its values to approach -1 in a similar way as $x \to -\infty$. *Now try Exercise 5.*

Sandwich Theorem Revisited

The Sandwich Theorem also holds for limits as $x \to \pm\infty$.



$[-4\pi, 4\pi]$ by $[-0.5, 1.5]$

(a)

X	Y1	
100	-.0051	
200	-.0044	
300	-.0033	
400	-.0021	
500	-9E-4	
600	7.4E-5	
700	7.8E-4	

Y1 = sin(X)/X

(b)

Figure 2.11 (a) The graph of $f(x) = (\sin x)/x$ oscillates about the x-axis. The amplitude of the oscillations decreases toward zero as $x \to \pm\infty$. (b) A table of values for f that suggests $f(x) \to 0$ as $x \to \infty$. (Example 2)

EXAMPLE 2 Finding a Limit as x Approaches ∞

Find $\lim_{x \to \infty} f(x)$ for $f(x) = \dfrac{\sin x}{x}$.

SOLUTION

Solve Graphically and Numerically The graph and table of values in Figure 2.11 suggest that $y = 0$ is the horizontal asymptote of f.

Confirm Analytically We know that $-1 \le \sin x \le 1$. So, for $x > 0$ we have

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}.$$

Therefore, by the Sandwich Theorem,

$$0 = \lim_{x \to \infty} \left(-\frac{1}{x}\right) = \lim_{x \to \infty} \frac{\sin x}{x} = \lim_{x \to \infty} \frac{1}{x} = 0.$$

Since $(\sin x)/x$ is an even function of x, we can also conclude that

$$\lim_{x \to \infty} \frac{\sin x}{x} = 0.$$

Now try Exercise 9.

Limits at infinity have properties similar to those of finite limits.

THEOREM 5 Properties of Limits as $x \to \pm\infty$

If L, M, and k are real numbers and

$$\lim_{x \to \pm\infty} f(x) = L \quad \text{and} \quad \lim_{x \to \pm\infty} g(x) = M, \text{ then}$$

1. *Sum Rule:* $\lim_{x \to \pm\infty} (f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to \pm\infty} (f(x) - g(x)) = L - M$
3. *Product Rule:* $\lim_{x \to \pm\infty} (f(x) \cdot g(x)) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{x \to \pm\infty} (k \cdot f(x)) = k \cdot L$
5. *Quotient Rule:* $\lim_{x \to \pm\infty} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, M \neq 0$
6. *Power Rule:* If r and s are integers, $s \neq 0$, then

$$\lim_{x \to \pm\infty} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number.

We can use Theorem 5 to find limits at infinity of functions with complicated expressions, as illustrated in Example 3.

EXAMPLE 3 Using Theorem 5

Find $\lim_{x\to\infty} \dfrac{5x + \sin x}{x}$.

SOLUTION

Notice that

$$\frac{5x + \sin x}{x} = \frac{5x}{x} + \frac{\sin x}{x} = 5 + \frac{\sin x}{x}.$$

So,

$$\lim_{x\to\infty} \frac{5x + \sin x}{x} = \lim_{x\to\infty} 5 + \lim_{x\to\infty} \frac{\sin x}{x} \quad \text{Sum Rule}$$

$$= 5 + 0 = 5. \quad \text{Known Values}$$

Now try Exercise 25.

EXPLORATION 1 Exploring Theorem 5

We must be careful how we apply Theorem 5.

1. (Example 3 again) Let $f(x) = 5x + \sin x$ and $g(x) = x$. Do the limits as $x\to\infty$ of f and g exist? Can we apply the Quotient Rule to $\lim_{x\to\infty} f(x)/g(x)$? Explain. Does the limit of the quotient exist?
2. Let $f(x) = \sin^2 x$ and $g(x) = \cos^2 x$. Describe the behavior of f and g as $x\to\infty$. Can we apply the Sum Rule to $\lim_{x\to\infty} (f(x) + g(x))$? Explain. Does the limit of the sum exist?
3. Let $f(x) = \ln(2x)$ and $g(x) = \ln(x + 1)$. Find the limits as $x\to\infty$ of f and g. Can we apply the Difference Rule to $\lim_{x\to\infty} (f(x) - g(x))$? Explain. Does the limit of the difference exist?
4. Based on parts 1–3, what advice might you give about applying Theorem 5?

Infinite Limits as $x\to a$

If the values of a function $f(x)$ outgrow all positive bounds as x approaches a finite number a, we say that $\lim_{x\to a} f(x) = \infty$. If the values of f become large and negative, exceeding all negative bounds as $x\to a$, we say that $\lim_{x\to a} f(x) = -\infty$.

Looking at $f(x) = 1/x$ (Figure 2.9, page 70), we observe that

$$\lim_{x\to 0^+} 1/x = \infty \quad \text{and} \quad \lim_{x\to 0^-} 1/x = -\infty.$$

We say that the line $x = 0$ is a *vertical asymptote* of the graph of f.

DEFINITION Vertical Asymptote

The line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty$$

EXAMPLE 4 Finding Vertical Asymptotes

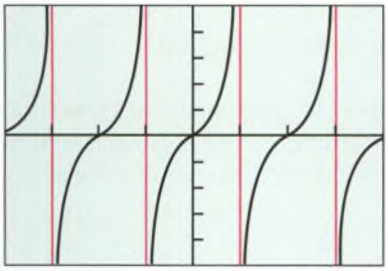
Find the vertical asymptotes of $f(x) = \dfrac{1}{x^2}$. Describe the behavior to the left and right of each vertical asymptote.

SOLUTION

The values of the function approach ∞ on either side of $x = 0$.

$$\lim_{x\to 0^+} \frac{1}{x^2} = \infty \quad \text{and} \quad \lim_{x\to 0^-} \frac{1}{x^2} = \infty.$$

The line $x = 0$ is the only vertical asymptote. *Now try Exercise 27.*



[-2π, 2π] by [-5, 5]

Figure 2.12 The graph of $f(x) = \tan x$ has a vertical asymptote at $\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$. (Example 5)

We can also say that $\lim_{x\to 0} (1/x^2) = \infty$. We can make no such statement about $1/x$.

EXAMPLE 5 Finding Vertical Asymptotes

The graph of $f(x) = \tan x = (\sin x)/(\cos x)$ has infinitely many vertical asymptotes, one at each point where the cosine is zero. If a is an odd multiple of $\pi/2$, then

$$\lim_{x\to a^+} \tan x = -\infty \quad \text{and} \quad \lim_{x\to a^-} \tan x = \infty,$$

as suggested by Figure 2.12. *Now try Exercise 31.*

You might think that the graph of a quotient always has a vertical asymptote where the denominator is zero, but that need not be the case. For example, we observed in Section 2.1 that $\lim_{x\to 0} (\sin x)/x = 1$.

End Behavior Models

For numerically large values of x, we can sometimes model the behavior of a complicated function by a simpler one that acts virtually in the same way.

EXAMPLE 6 Modeling Functions For $|x|$ Large

Let $f(x) = 3x^4 - 2x^3 + 3x^2 - 5x + 6$ and $g(x) = 3x^4$. Show that while f and g are quite different for numerically small values of x, they are virtually identical for $|x|$ large.

SOLUTION

Solve Graphically The graphs of f and g (Figure 2.13a), quite different near the origin, are virtually identical on a larger scale (Figure 2.13b).

Confirm Analytically We can test the claim that g models f for numerically large values of x by examining the ratio of the two functions as $x\to\pm\infty$. We find that

$$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \lim_{x\to\pm\infty} \frac{3x^4 - 2x^3 + 3x^2 - 5x + 6}{3x^4}$$
$$= \lim_{x\to\pm\infty} \left(1 - \frac{2}{3x} + \frac{1}{x^2} - \frac{5}{3x^3} + \frac{2}{x^4}\right)$$
$$= 1,$$

convincing evidence that f and g behave alike for $|x|$ large. *Now try Exercise 39.*

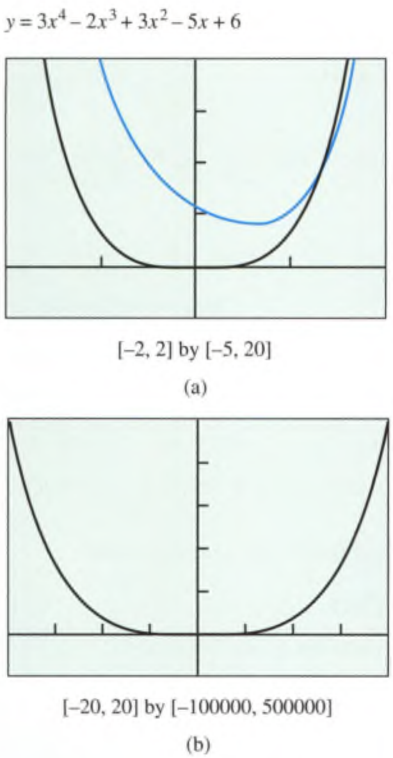


Figure 2.13 The graphs of f and g, (a) distinct for $|x|$ small, are (b) nearly identical for $|x|$ large. (Example 6)

DEFINITION End Behavior Model

The function g is

(a) a **right end behavior model** for f if and only if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

(b) a **left end behavior model** for f if and only if $\lim_{x\to-\infty} \frac{f(x)}{g(x)} = 1$.

If one function provides both a left and right end behavior model, it is simply called an **end behavior model.** Thus, $g(x) = 3x^4$ is an end behavior model for $f(x) = 3x^4 - 2x^3 + 3x^2 - 5x + 6$ (Example 6).

In general, $g(x) = a_n x^n$ is an end behavior model for the polynomial function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, $a_n \neq 0$. Overall, the end behavior of all polynomials behave like the end behavior of monomials. This is the key to the end behavior of rational functions, as illustrated in Example 7.

EXAMPLE 7 Finding End Behavior Models

Find an end behavior model for

(a) $f(x) = \dfrac{2x^5 + x^4 - x^2 + 1}{3x^2 - 5x + 7}$ **(b)** $g(x) = \dfrac{2x^3 - x^2 + x - 1}{5x^3 + x^2 + x - 5}$

SOLUTION

(a) Notice that $2x^5$ is an end behavior model for the numerator of f, and $3x^2$ is one for the denominator. This makes

$$\frac{2x^5}{3x^2} = \frac{2}{3}x^3$$

an end behavior model for f.

(b) Similarly, $2x^3$ is an end behavior model for the numerator of g, and $5x^3$ is one for the denominator of g. This makes

$$\frac{2x^3}{5x^3} = \frac{2}{5}$$

an end behavior model for g. *Now try Exercise 43.*

Notice in Example 7b that the end behavior model for g, $y = 2/5$, is also a horizontal asymptote of the graph of g, while in 7a, the graph of f does not have a horizontal asymptote. We can use the end behavior model of a rational function to identify any horizontal asymptote.

We can see from Example 7 that a rational function always has a simple power function as an end behavior model.

A function's right and left end behavior models need not be the same function.

EXAMPLE 8 Finding End Behavior Models

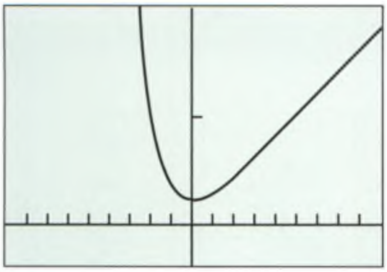
Let $f(x) = x + e^{-x}$. Show that $g(x) = x$ is a right end behavior model for f while $h(x) = e^{-x}$ is a left end behavior model for f.

SOLUTION

On the right,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x + e^{-x}}{x} = \lim_{x\to\infty} \left(1 + \frac{e^{-x}}{x}\right) = 1 \text{ because } \lim_{x\to\infty} \frac{e^{-x}}{x} = 0.$$

continued



[–9, 9] by [–2, 10]

Figure 2.14 The graph of $f(x) = x + e^{-x}$ looks like the graph of $g(x) = x$ to the right of the y-axis, and like the graph of $h(x) = e^{-x}$ to the left of the y-axis. (Example 8)

On the left,

$$\lim_{x\to-\infty} \frac{f(x)}{h(x)} = \lim_{x\to-\infty} \frac{x + e^{-x}}{e^{-x}} = \lim_{x\to-\infty} \left(\frac{x}{e^{-x}} + 1\right) = 1 \text{ because } \lim_{x\to-\infty} \frac{x}{e^{-x}} = 0.$$

The graph of f in Figure 2.14 supports these end behavior conclusions.

Now try Exercise 45.

"Seeing" Limits as $x\to\pm\infty$

We can investigate the graph of $y = f(x)$ as $x\to\pm\infty$ by investigating the graph of $y = f(1/x)$ as $x\to 0$.

EXAMPLE 9 Using Substitution

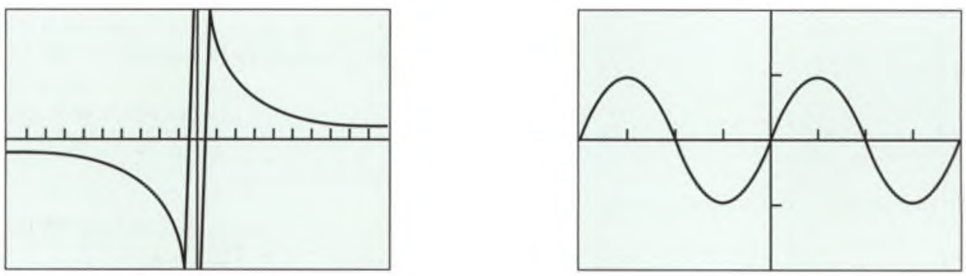
Find $\lim_{x\to\infty} \sin(1/x)$.

SOLUTION

Figure 2.15a suggests that the limit is 0. Indeed, replacing $\lim_{x\to\infty} \sin(1/x)$ by the equivalent $\lim_{x\to 0^+} \sin x = 0$ (Figure 2.15b), we find

$$\lim_{x\to\infty} \sin 1/x = \lim_{x\to 0^+} \sin x = 0.$$

Now try Exercise 49.



[–10, 10] by [–1, 1]

(a)

[-2π, 2π] by [–2, 2]

(b)

Figure 2.15 The graphs of (a) $f(x) = \sin(1/x)$ and (b) $g(x) = f(1/x) = \sin x$. (Example 9)

Quick Review 2.2 *(For help, go to Section 1.2 and 1.5.)*

In Exercises 1–4, find f^{-1} and graph f, f^{-1}, and $y = x$ in the same square viewing window.

1. $f(x) = 2x - 3$

2. $f(x) = e^x$

3. $f(x) = \tan^{-1} x$

4. $f(x) = \cot^{-1} x$

In Exercises 5 and 6, find the quotient $q(x)$ and remainder $r(x)$ when $f(x)$ is divided by $g(x)$.

5. $f(x) = 2x^3 - 3x^2 + x - 1$, $g(x) = 3x^3 + 4x - 5$

6. $f(x) = 2x^5 - x^3 + x - 1$, $g(x) = x^3 - x^2 + 1$

In Exercises 7–10, write a formula for **(a)** $f(-x)$ and **(b)** $f(1/x)$. Simplify where possible.

7. $f(x) = \cos x$

8. $f(x) = e^{-x}$

9. $f(x) = \dfrac{\ln x}{x}$

10. $f(x) = \left(x + \dfrac{1}{x}\right)\sin x$

Section 2.2 Exercises

In Exercises 1–8, use graphs and tables to find **(a)** $\lim_{x\to\infty} f(x)$ and **(b)** $\lim_{x\to-\infty} f(x)$ **(c)** Identify all horizontal asymptotes.

1. $f(x) = \cos\left(\dfrac{1}{x}\right)$ **2.** $f(x) = \dfrac{\sin 2x}{x}$

3. $f(x) = \dfrac{e^{-x}}{x}$ **4.** $f(x) = \dfrac{3x^3 - x + 1}{x + 3}$

5. $f(x) = \dfrac{3x + 1}{|x| + 2}$ **6.** $f(x) = \dfrac{2x - 1}{|x| - 3}$

7. $f(x) = \dfrac{x}{|x|}$ **8.** $f(x) = \dfrac{|x|}{|x| + 1}$

In Exercises 9–12, find the limit and confirm your answer using the Sandwich Theorem.

9. $\displaystyle\lim_{x\to\infty} \frac{1 - \cos x}{x^2}$ **10.** $\displaystyle\lim_{x\to-\infty} \frac{1 - \cos x}{x^2}$

11. $\displaystyle\lim_{x\to-\infty} \frac{\sin x}{x}$ **12.** $\displaystyle\lim_{x\to\infty} \frac{\sin (x^2)}{x}$

In Exercises 13–20, use graphs and tables to find the limits.

13. $\displaystyle\lim_{x\to2^+} \frac{1}{x - 2}$ **14.** $\displaystyle\lim_{x\to2^-} \frac{x}{x - 2}$

15. $\displaystyle\lim_{x\to-3^-} \frac{1}{x + 3}$ **16.** $\displaystyle\lim_{x\to-3^+} \frac{x}{x + 3}$

17. $\displaystyle\lim_{x\to0^+} \frac{\text{int } x}{x}$ **18.** $\displaystyle\lim_{x\to0^-} \frac{\text{int } x}{x}$

19. $\displaystyle\lim_{x\to0^+} \csc x$ **20.** $\displaystyle\lim_{x\to(\pi/2)^+} \sec x$

In Exercises 21–26, find $\lim_{x\to\infty} y$ and $\lim_{x\to-\infty} y$.

21. $y = \left(2 - \dfrac{x}{x + 1}\right)\left(\dfrac{x^2}{5 + x^2}\right)$ **22.** $y = \left(\dfrac{2}{x} + 1\right)\left(\dfrac{5x^2 - 1}{x^2}\right)$

23. $y = \dfrac{\cos (1/x)}{1 + (1/x)}$ **24.** $y = \dfrac{2x + \sin x}{x}$

25. $y = \dfrac{\sin x}{2x^2 + x}$ **26.** $y = \dfrac{x \sin x + 2 \sin x}{2x^2}$

In Exercises 27–34, **(a)** find the vertical asymptotes of the graph of $f(x)$. **(b)** Describe the behavior of $f(x)$ to the left and right of each vertical asymptote.

27. $f(x) = \dfrac{1}{x^2 - 4}$ **28.** $f(x) = \dfrac{x^2 - 1}{2x + 4}$

29. $f(x) = \dfrac{x^2 - 2x}{x + 1}$ **30.** $f(x) = \dfrac{1 - x}{2x^2 - 5x - 3}$

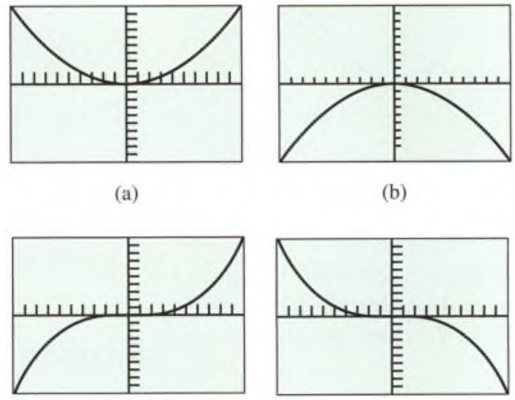
31. $f(x) = \cot x$ **32.** $f(x) = \sec x$

33. $f(x) = \dfrac{\tan x}{\sin x}$ **34.** $f(x) = \dfrac{\cot x}{\cos x}$

In Exercises 35–38, match the function with the graph of its end behavior model.

35. $y = \dfrac{2x^3 - 3x^2 + 1}{x + 3}$ **36.** $y = \dfrac{x^5 - x^4 + x + 1}{2x^2 + x - 3}$

37. $y = \dfrac{2x^4 - x^3 + x^2 - 1}{2 - x}$ **38.** $y = \dfrac{x^4 - 3x^3 + x^2 - 1}{1 - x^2}$



(a) (b) (c) (d)

In Exercises 39–44, **(a)** find a power function end behavior model for f. **(b)** Identify any horizontal asymptotes.

39. $f(x) = 3x^2 - 2x + 1$ **40.** $f(x) = -4x^3 + x^2 - 2x - 1$

41. $f(x) = \dfrac{x - 2}{2x^2 + 3x - 5}$ **42.** $f(x) = \dfrac{3x^2 - x + 5}{x^2 - 4}$

43. $f(x) = \dfrac{4x^3 - 2x + 1}{x - 2}$ **44.** $f(x) = \dfrac{-x^4 + 2x^2 + x - 3}{x^2 - 4}$

In Exercises 45–48, find **(a)** a simple basic function as a right end behavior model and **(b)** a simple basic function as a left end behavior model for the function.

45. $y = e^x - 2x$ **46.** $y = x^2 + e^{-x}$

47. $y = x + \ln |x|$ **48.** $y = x^2 + \sin x$

In Exercises 49–52, use the graph of $y = f(1/x)$ to find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

49. $f(x) = xe^x$ **50.** $f(x) = x^2e^{-x}$

51. $f(x) = \dfrac{\ln |x|}{x}$ **52.** $f(x) = x \sin \dfrac{1}{x}$

In Exercises 53 and 54, find the limit of $f(x)$ as **(a)** $x\to-\infty$, **(b)** $x\to\infty$, **(c)** $x\to0^-$, and **(d)** $x\to0^+$.

53. $f(x) = \begin{cases} 1/x, & x < 0 \\ -1, & x \ge 0 \end{cases}$

54. $f(x) = \begin{cases} \dfrac{x - 2}{x - 1}, & x \le 0 \\ 1/x^2, & x > 0 \end{cases}$

Group Activity In Exercises 55 and 56, sketch a graph of a function $y = f(x)$ that satisfies the stated conditions. Include any asymptotes.

55. $\displaystyle\lim_{x\to1} f(x) = 2,\ \lim_{x\to5^-} f(x) = \infty,\ \lim_{x\to5^+} f(x) = \infty,$

$\displaystyle\lim_{x\to\infty} f(x) = -1,\ \lim_{x\to-2^+} f(x) = -\infty,$

$\displaystyle\lim_{x\to-2^-} f(x) = \infty,\ \lim_{x\to-\infty} f(x) = 0$

56. $\displaystyle\lim_{x\to2} f(x) = -1,\ \lim_{x\to4^+} f(x) = -\infty,\ \lim_{x\to4^-} f(x) = \infty,$

$\displaystyle\lim_{x\to\infty} f(x) = \infty,\ \lim_{x\to-\infty} f(x) = 2$

57. Group Activity *End Behavior Models* Suppose that $g_1(x)$ is a right end behavior model for $f_1(x)$ and that $g_2(x)$ is a right end behavior model for $f_2(x)$. Explain why this makes $g_1(x)/g_2(x)$ a right end behavior model for $f_1(x)/f_2(x)$.

58. Writing to Learn Let L be a real number, $\lim_{x\to c} f(x) = L$, and $\lim_{x\to c} g(x) = \infty$ or $-\infty$. Can $\lim_{x\to c} (f(x) + g(x))$ be determined? Explain.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

59. True or False It is possible for a function to have more than one horizontal asymptote. Justify your answer.

60. True or False If $f(x)$ has a vertical asymptote at $x = c$, then either $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = \infty$ or $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = -\infty$. Justify your answer.

61. Multiple Choice $\lim_{x\to 2^-} \dfrac{x}{x-2} =$

(A) $-\infty$ **(B)** ∞ **(C)** 1 **(D)** $-1/2$ **(E)** -1

62. Multiple Choice $\lim_{x\to 0} \dfrac{\cos(2x)}{x} =$

(A) 1/2 **(B)** 1 **(C)** 2 **(D)** cos 2 **(E)** does not exist

63. Multiple Choice $\lim_{x\to 0} \dfrac{\sin(3x)}{x} =$

(A) 1/3 **(B)** 1 **(C)** 3 **(D)** sin 3 **(E)** does not exist

64. Multiple Choice Which of the following is an end behavior for

$$f(x) = \frac{2x^3 - x^2 + x + 1}{x^3 - 1}?$$

(A) x^3 **(B)** $2x^3$ **(C)** $1/x^3$ **(D)** 2 **(E)** 1/2

Exploration

65. *Exploring Properties of Limits* Find the limits of f, g, and fg as $x \to c$.

(a) $f(x) = \dfrac{1}{x}$, $g(x) = x$, $c = 0$

(b) $f(x) = -\dfrac{2}{x^3}$, $g(x) = 4x^3$, $c = 0$

(c) $f(x) = \dfrac{3}{x-2}$, $g(x) = (x-2)^3$, $c = 2$

(d) $f(x) = \dfrac{5}{(3-x)^4}$, $g(x) = (x-3)^2$, $c = 3$

(e) Writing to Learn Suppose that $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = \infty$. Based on your observations in parts (a)–(d), what can you say about $\lim_{x\to c} (f(x) \cdot g(x))$?

Extending the Ideas

66. *The Greatest Integer Function*

(a) Show that

$$\frac{x-1}{x} < \frac{\text{int } x}{x} \le 1 \ (x > 0) \text{ and } \frac{x-1}{x} > \frac{\text{int } x}{x} \ge 1 \ (x < 0).$$

(b) Determine $\lim_{x\to\infty} \dfrac{\text{int } x}{x}$.

(c) Determine $\lim_{x\to -\infty} \dfrac{\text{int } x}{x}$.

67. *Sandwich Theorem* Use the Sandwich Theorem to confirm the limit as $x \to \infty$ found in Exercise 3.

68. Writing to Learn Explain why there is no value L for which $\lim_{x\to\infty} \sin x = L$.

In Exercises 69–71, find the limit. Give a convincing argument that the value is correct.

69. $\lim_{x\to\infty} \dfrac{\ln x^2}{\ln x}$

70. $\lim_{x\to\infty} \dfrac{\ln x}{\log x}$

71. $\lim_{x\to\infty} \dfrac{\ln (x+1)}{\ln x}$

Quick Quiz for AP* Preparation: Sections 2.1 and 2.2

You should solve the following problems without using a graphing calculator.

1. Multiple Choice Find $\lim_{x\to 3} \dfrac{x^2 - x - 6}{x - 3}$, if it exists.

(A) -1 **(B)** 1 **(C)** 2 **(D)** 5 **(E)** does not exist

2. Multiple Choice Find $\lim_{x\to 2^+} f(x)$, if it exists, where

$$f(x) = \begin{cases} 3x + 1, & x < 2 \\ \dfrac{5}{x+1}, & x \ge 2 \end{cases}$$

(A) 5/3 **(B)** 13/3 **(C)** 7 **(D)** ∞ **(E)** does not exist

3. Multiple Choice Which of the following lines is a horizontal asymptote for

$$f(x) = \frac{3x^3 - x^2 + x - 7}{2x^3 + 4x - 5}?$$

(A) $y = \dfrac{3}{2}x$ **(B)** $y = 0$ **(C)** $y = 2/3$ **(D)** $y = 7/5$ **(E)** $y = 3/2$

4. Free Response Let $f(x) = \dfrac{\cos x}{x}$.

(a) Find the domain and range of f.

(b) Is f even, odd, or neither? Justify your answer.

(c) Find $\lim_{x\to\infty} f(x)$.

(d) Use the Sandwich Theorem to justify your answer to part (c).

2.3 Continuity

What you'll learn about

- Continuity at a Point
- Continuous Functions
- Algebraic Combinations
- Composites
- Intermediate Value Theorem for Continuous Functions

. . . and why

Continuous functions are used to describe how a body moves through space and how the speed of a chemical reaction changes with time.

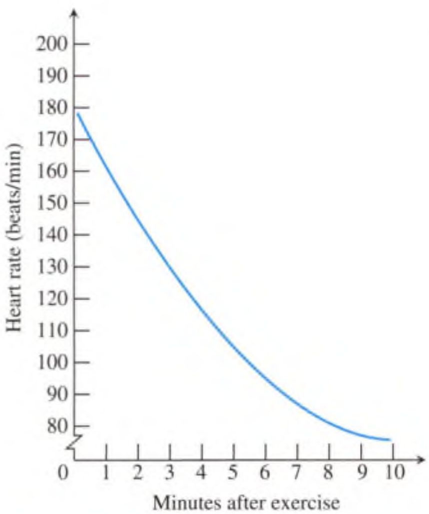


Figure 2.16 How the heartbeat returns to a normal rate after running.

Continuity at a Point

When we plot function values generated in the laboratory or collected in the field, we often connect the plotted points with an unbroken curve to show what the function's values are likely to have been at the times we did not measure (Figure 2.16). In doing so, we are assuming that we are working with a *continuous function,* a function whose outputs vary continuously with the inputs and do not jump from one value to another without taking on the values in between. Any function $y = f(x)$ whose graph can be sketched in one continuous motion without lifting the pencil is an example of a continuous function.

Continuous functions are the functions we use to find a planet's closest point of approach to the sun or the peak concentration of antibodies in blood plasma. They are also the functions we use to describe how a body moves through space or how the speed of a chemical reaction changes with time. In fact, so many physical processes proceed continuously that throughout the eighteenth and nineteenth centuries it rarely occurred to anyone to look for any other kind of behavior. It came as a surprise when the physicists of the 1920s discovered that light comes in particles and that heated atoms emit light at discrete frequencies (Figure 2.17). As a result of these and other discoveries, and because of the heavy use of discontinuous functions in computer science, statistics, and mathematical modeling, the issue of continuity has become one of practical as well as theoretical importance.

To understand continuity, we need to consider a function like the one in Figure 2.18, whose limits we investigated in Example 8, Section 2.1.

Figure 2.17 The laser was developed as a result of an understanding of the nature of the atom.

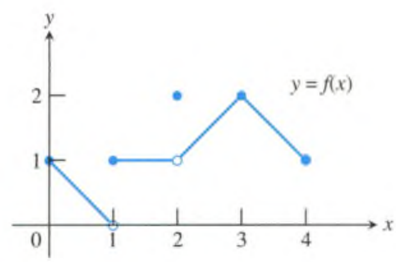


Figure 2.18 The function is continuous on [0, 4] except at $x = 1$ and $x = 2$. (Example 1)

EXAMPLE 1 Investigating Continuity

Find the points at which the function f in Figure 2.18 is continuous, and the points at which f is discontinuous.

SOLUTION

The function f is continuous at every point in its domain [0, 4] except at $x = 1$ and $x = 2$. At these points there are breaks in the graph. Note the relationship between the limit of f and the value of f at each point of the function's domain.

Points at which f is continuous:

At $x = 0$, $\quad \lim_{x \to 0^+} f(x) = f(0).$

At $x = 4$, $\quad \lim_{x \to 4^-} f(x) = f(4).$

At $0 < c < 4,\ c \neq 1, 2,$ $\quad \lim_{x \to c} f(x) = f(c).$

continued

Points at which f is discontinuous:

At $x = 1$, $\quad \lim_{x \to 1} f(x)$ does not exist.

At $x = 2$, $\quad \lim_{x \to 2} f(x) = 1$, but $1 \neq f(2)$.

At $c < 0, c > 4$, $\quad$ these points are not in the domain of f.

Now try Exercise 5.

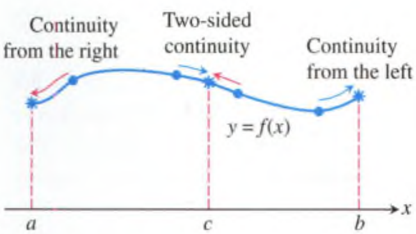


Figure 2.19 Continuity at points a, b, and c for a function $y = f(x)$ that is continuous on the interval $[a, b]$.

To define continuity at a point in a function's domain, we need to define continuity at an interior point (which involves a two-sided limit) and continuity at an endpoint (which involves a one-sided limit). (Figure 2.19)

DEFINITION Continuity at a Point

Interior Point: A function $y = f(x)$ is **continuous at an interior point *c*** of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint *a*** or is **continuous at a right endpoint *b*** of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively.}$$

If a function f is not continuous at a point c, we say that f is **discontinuous** at c and c is a **point of discontinuity** of f. Note that c need not be in the domain of f.

EXAMPLE 2 Finding Points of Continuity and Discontinuity

Find the points of continuity and the points of discontinuity of the greatest integer function (Figure 2.20).

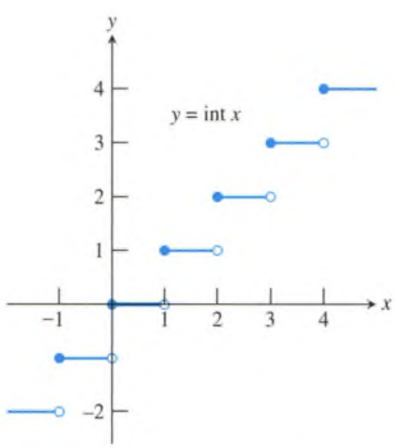


Figure 2.20 The function int x is continuous at every noninteger point. (Example 2)

SOLUTION

For the function to be continuous at $x = c$, the limit as $x \to c$ must exist and must equal the value of the function at $x = c$. The greatest integer function is discontinuous at every integer. For example,

$$\lim_{x \to 3^-} \text{int } x = 2 \quad \text{and} \quad \lim_{x \to 3^+} \text{int } x = 3$$

so the limit as $x \to 3$ does not exist. Notice that int $3 = 3$. In general, if n is any integer,

$$\lim_{x \to n^-} \text{int } x = n - 1 \quad \text{and} \quad \lim_{x \to n^+} \text{int } x = n,$$

so the limit as $x \to n$ does not exist.

The greatest integer function is continuous at every other real number. For example,

$$\lim_{x \to 1.5} \text{int } x = 1 = \text{int } 1.5.$$

In general, if $n - 1 < c < n$, n an integer, then

$$\lim_{x \to c} \text{int } x = n - 1 = \text{int } c.$$

Now try Exercise 7.

Shirley Ann Jackson
(1946–)

Distinguished scientist, Shirley Jackson credits her interest in science to her parents and excellent mathematics and science teachers in high school. She studied physics, and in 1973, became the first African American woman to earn a Ph.D. at the Massachusetts Institute of Technology. Since then, Dr. Jackson has done research on topics relating to theoretical material sciences, has received numerous scholarships and honors, and has published more than one hundred scientific articles.

Figure 2.21 is a catalog of discontinuity types. The function in (a) is continuous at $x = 0$. The function in (b) would be continuous if it had $f(0) = 1$. The function in (c) would be continuous if $f(0)$ were 1 instead of 2. The discontinuities in (b) and (c) are **removable.** Each function has a limit as $x \to 0$, and we can remove the discontinuity by setting $f(0)$ equal to this limit.

The discontinuities in (d)–(f) of Figure 2.21 are more serious: $\lim_{x \to 0} f(x)$ does not exist and there is no way to improve the situation by changing f at 0. The step function in (d) has a **jump discontinuity:** the one-sided limits exist but have different values. The function $f(x) = 1/x^2$ in (e) has an **infinite discontinuity.** The function in (f) has an **oscillating discontinuity:** it oscillates and has no limit as $x \to 0$.

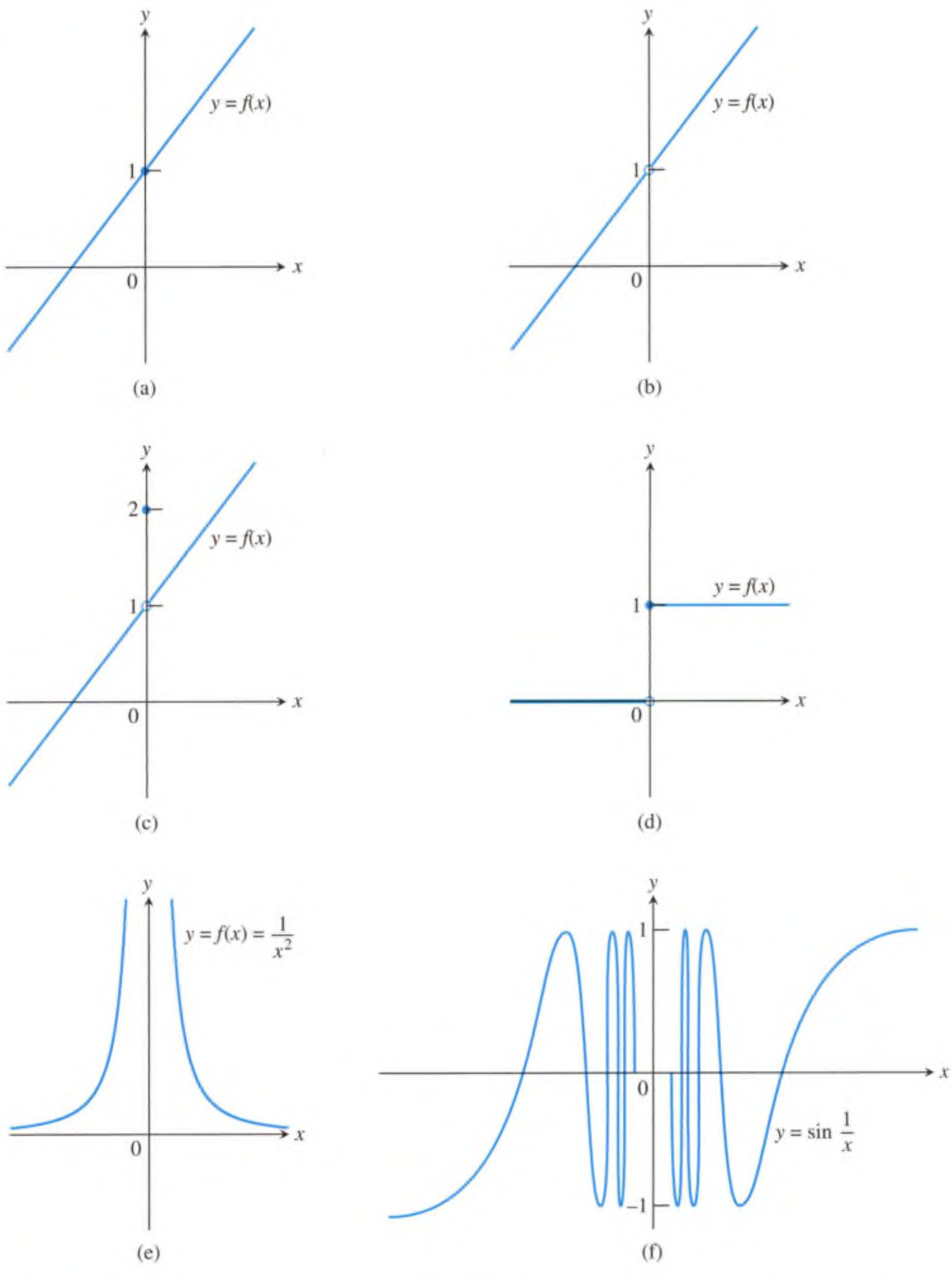


Figure 2.21 The function in part (a) is continuous at $x = 0$. The functions in parts (b)–(f) are not.

EXPLORATION 1 **Removing a Discontinuity**

Let $f(x) = \dfrac{x^3 - 7x - 6}{x^2 - 9}$.

1. Factor the denominator. What is the domain of f?
2. Investigate the graph of f around $x = 3$ to see that f has a removable discontinuity at $x = 3$.
3. How should f be defined at $x = 3$ to remove the discontinuity? Use zoom-in and tables as necessary.
4. Show that $(x - 3)$ is a factor of the numerator of f, and remove all common factors. Now compute the limit as $x \to 3$ of the reduced form for f.
5. Show that the *extended function*

$$g(x) = \begin{cases} \dfrac{x^3 - 7x - 6}{x^2 - 9}, & x \neq 3 \\ 10/3, & x = 3 \end{cases}$$

is continuous at $x = 3$. The function g is the **continuous extension** of the original function f to include $x = 3$.

Now try Exercise 25.

Continuous Functions

A function is **continuous on an interval** if and only if it is continuous at every point of the interval. A **continuous function** is one that is continuous at every point of its domain. A continuous function need not be continuous on every interval. For example, $y = 1/x$ is not continuous on $[-1, 1]$.

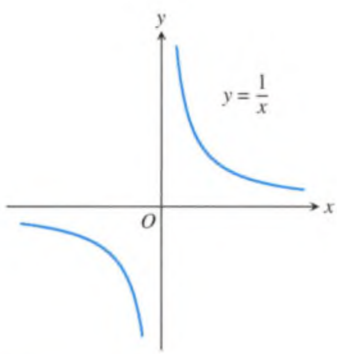


Figure 2.22 The function $y = 1/x$ is continuous at every value of x except $x = 0$. It has a point of discontinuity at $x = 0$. (Example 3)

EXAMPLE 3 Identifying Continuous Functions

The reciprocal function $y = 1/x$ (Figure 2.22) is a continuous function because it is continuous at every point of its domain. However, it has a point of discontinuity at $x = 0$ because it is not defined there.

Now try Exercise 31.

Polynomial functions f are continuous at every real number c because $\lim_{x \to c} f(x) = f(c)$. Rational functions are continuous at every point of their domains. They have points of discontinuity at the zeros of their denominators. The absolute value function $y = |x|$ is continuous at every real number. The exponential functions, logarithmic functions, trigonometric functions, and radical functions like $y = \sqrt[n]{x}$ (n a positive integer greater than 1) are continuous at every point of their domains. All of these functions are continuous functions.

Algebraic Combinations

As you may have guessed, algebraic combinations of continuous functions are continuous wherever they are defined.

THEOREM 6 Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums:* $f + g$
2. *Differences:* $f - g$
3. *Products:* $f \cdot g$
4. *Constant multiples:* $k \cdot f$, for any number k
5. *Quotients:* f/g, provided $g(c) \neq 0$

Composites

All composites of continuous functions are continuous. This means composites like

$$y = \sin(x^2) \quad \text{and} \quad y = |\cos x|$$

are continuous at every point at which they are defined. The idea is that if $f(x)$ is continuous at $x = c$ and $g(x)$ is continuous at $x = f(c)$, then $g \circ f$ is continuous at $x = c$ (Figure 2.23). In this case, the limit as $x \to c$ is $g(f(c))$.

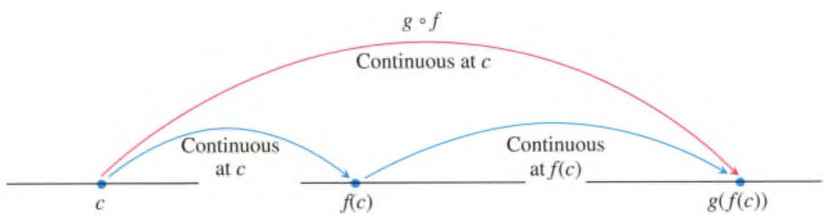


Figure 2.23 Composites of continuous functions are continuous.

THEOREM 7 Composite of Continuous Functions

If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.

EXAMPLE 4 Using Theorem 7

Show that $y = \left|\dfrac{x \sin x}{x^2 + 2}\right|$ is continuous.

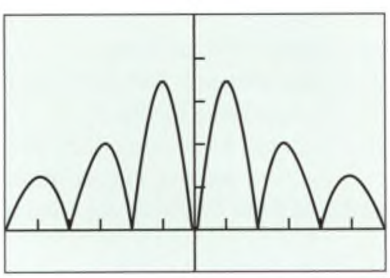
SOLUTION

The graph (Figure 2.24) of $y = |(x \sin x)/(x^2 + 2)|$ suggests that the function is continuous at every value of x. By letting

$$g(x) = |x| \quad \text{and} \quad f(x) = \frac{x \sin x}{x^2 + 2},$$

we see that y is the composite $g \circ f$.

We know that the absolute value function g is continuous. The function f is continuous by Theorem 6. Their composite is continuous by Theorem 7. ***Now try Exercise 33.***



$[-3\pi, 3\pi]$ by $[-0.1, 0.5]$

Figure 2.24 The graph suggests that $y = |(x \sin x)/(x^2 + 2)|$ is continuous. (Example 4)

Intermediate Value Theorem for Continuous Functions

Functions that are continuous on intervals have properties that make them particularly useful in mathematics and its applications. One of these is the *intermediate value property.* A function is said to have the **intermediate value property** if it never takes on two values without taking on all the values in between.

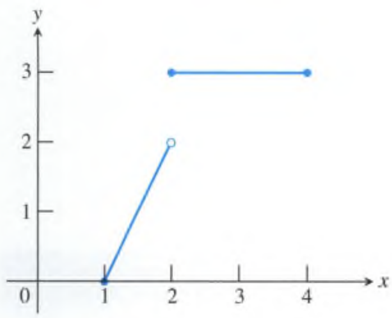


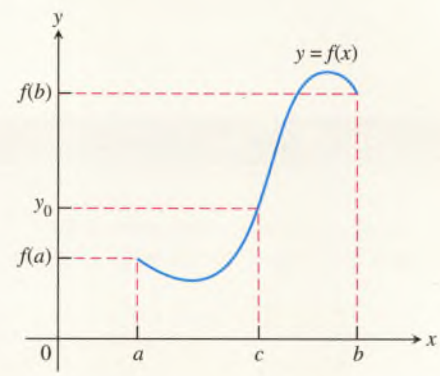
Figure 2.25 The function

$$f(x) = \begin{cases} 2x - 2, & 1 \le x < 2 \\ 3, & 2 \le x \le 4 \end{cases}$$

does not take on all values between $f(1) = 0$ and $f(4) = 3$; it misses all the values between 2 and 3.

THEOREM 8 The Intermediate Value Theorem for Continuous Functions

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



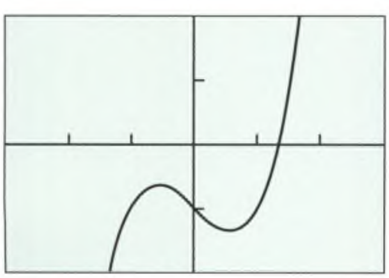
Grapher Failure

In connected mode, a grapher may conceal a function's discontinuities by portraying the graph as a connected curve when it is not. To see what we mean, graph $y = \text{int}(x)$ in a $[-10, 10]$ by $[-10, 10]$ window in both connected and dot modes. A knowledge of where to expect discontinuities will help you recognize this form of grapher failure.

The continuity of f on the interval is essential to Theorem 8. If f is discontinuous at even one point of the interval, the theorem's conclusion may fail, as it does for the function graphed in Figure 2.25.

A Consequence for Graphing: Connectivity Theorem 8 is the reason why the graph of a function continuous on an interval cannot have any breaks. The graph will be **connected,** a single, unbroken curve, like the graph of sin x. It will not have jumps like those in the graph of the greatest integer function int x, or separate branches like we see in the graph of $1/x$.

Most graphers can plot points *(dot mode)*. Some can turn on pixels between plotted points to suggest an unbroken curve *(connected mode)*. For functions, the connected format basically assumes that outputs *vary continuously* with inputs and do not jump from one value to another without taking on all values in between.



[–3, 3] by [–2, 2]

Figure 2.26 The graph of $f(x) = x^3 - x - 1$. (Example 5)

EXAMPLE 5 Using Theorem 8

Is any real number exactly 1 less than its cube?

SOLUTION

We answer this question by applying the Intermediate Value Theorem in the following way. Any such number must satisfy the equation $x = x^3 - 1$ or, equivalently, $x^3 - x - 1 = 0$. Hence, we are looking for a zero value of the continuous function $f(x) = x^3 - x - 1$ (Figure 2.26). The function changes sign between 1 and 2, so there must be a point c between 1 and 2 where $f(c) = 0$.

Now try Exercise 46.

Quick Review 2.3 *(For help, go to Sections 1.2 and 2.1.)*

1. Find $\lim_{x\to -1} \dfrac{3x^2 - 2x + 1}{x^3 + 4}$.

2. Let $f(x) = \text{int}\, x$. Find each limit.

(a) $\lim_{x\to -1^-} f(x)$ **(b)** $\lim_{x\to -1^+} f(x)$ **(c)** $\lim_{x\to -1} f(x)$ **(d)** $f(-1)$

3. Let $f(x) = \begin{cases} x^2 - 4x + 5, & x < 2 \\ 4 - x, & x \geq 2. \end{cases}$

Find each limit.

(a) $\lim_{x\to 2^-} f(x)$ **(b)** $\lim_{x\to 2^+} f(x)$ **(c)** $\lim_{x\to 2} f(x)$ **(d)** $f(2)$

In Exercises 4–6, find the remaining functions in the list of functions: $f, g, f \circ g, g \circ f$.

4. $f(x) = \dfrac{2x - 1}{x + 5}$, $g(x) = \dfrac{1}{x} + 1$

5. $f(x) = x^2$, $(g \circ f)(x) = \sin x^2$, domain of $g = [0, \infty)$

6. $g(x) = \sqrt{x - 1}$, $(g \circ f)(x) = 1/x$, $x > 0$

7. Use factoring to solve $2x^2 + 9x - 5 = 0$.

8. Use graphing to solve $x^3 + 2x - 1 = 0$.

In Exercises 9 and 10, let

$$f(x) = \begin{cases} 5 - x, & x \leq 3 \\ -x^2 + 6x - 8, & x > 3. \end{cases}$$

9. Solve the equation $f(x) = 4$.

10. Find a value of c for which the equation $f(x) = c$ has no solution.

Section 2.3 Exercises

In Exercises 1–10, find the points of continuity and the points of discontinuity of the function. Identify each type of discontinuity.

1. $y = \dfrac{1}{(x + 2)^2}$

2. $y = \dfrac{x + 1}{x^2 - 4x + 3}$

3. $y = \dfrac{1}{x^2 + 1}$

4. $y = |x - 1|$

5. $y = \sqrt{2x + 3}$

6. $y = \sqrt[3]{2x - 1}$

7. $y = |x|/x$

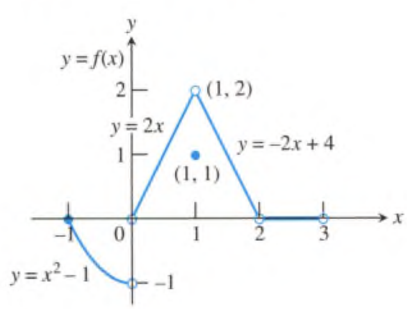
8. $y = \cot x$

9. $y = e^{1/x}$

10. $y = \ln (x + 1)$

In Exercises 11–18, use the function f defined and graphed below to answer the questions.

$$f(x) = \begin{cases} x^2 - 1, & -1 \leq x < 0 \\ 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -2x + 4, & 1 < x < 2 \\ 0, & 2 < x < 3 \end{cases}$$



11. (a) Does $f(-1)$ exist?

(b) Does $\lim_{x\to -1^+} f(x)$ exist?

(c) Does $\lim_{x\to -1^+} f(x) = f(-1)$?

(d) Is f continuous at $x = -1$?

12. (a) Does $f(1)$ exist?

(b) Does $\lim_{x\to 1} f(x)$ exist?

(c) Does $\lim_{x\to 1} f(x) = f(1)$?

(d) Is f continuous at $x = 1$?

13. (a) Is f defined at $x = 2$? (Look at the definition of f.)

(b) Is f continuous at $x = 2$?

14. At what values of x is f continuous?

15. What value should be assigned to $f(2)$ to make the extended function continuous at $x = 2$?

16. What new value should be assigned to $f(1)$ to make the new function continuous at $x = 1$?

17. Writing to Learn Is it possible to extend f to be continuous at $x = 0$? If so, what value should the extended function have there? If not, why not?

18. Writing to Learn Is it possible to extend f to be continuous at $x = 3$? If so, what value should the extended function have there? If not, why not?

In Exercises 19–24, **(a)** find each point of discontinuity. **(b)** Which of the discontinuities are removable? not removable? Give reasons for your answers.

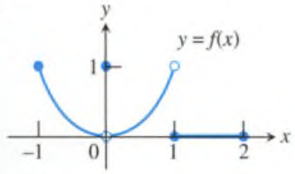
19. $f(x) = \begin{cases} 3 - x, & x < 2 \\ \dfrac{x}{2} + 1, & x > 2 \end{cases}$

20. $f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ x/2, & x > 2 \end{cases}$

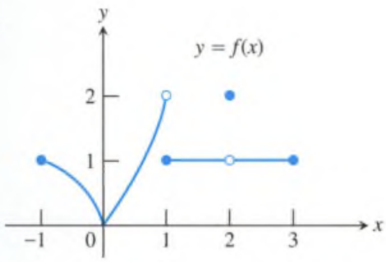
21. $f(x) = \begin{cases} \dfrac{1}{x - 1}, & x < 1 \\ x^3 - 2x + 5, & x \geq 1 \end{cases}$

22. $f(x) = \begin{cases} 1 - x^2, & x \neq -1 \\ 2, & x = -1 \end{cases}$

23.



24.



In Exercises 25–30, give a formula for the extended function that is continuous at the indicated point.

25. $f(x) = \dfrac{x^2 - 9}{x + 3}, \quad x = -3$

26. $f(x) = \dfrac{x^3 - 1}{x^2 - 1}, \quad x = 1$

27. $f(x) = \dfrac{\sin x}{x}, \quad x = 0$

28. $f(x) = \dfrac{\sin 4x}{x}, \quad x = 0$

29. $f(x) = \dfrac{x - 4}{\sqrt{x} - 2}, \quad x = 4$

30. $f(x) = \dfrac{x^3 - 4x^2 - 11x + 30}{x^2 - 4}, \quad x = 2$

In Exercises 31 and 32, explain why the given function is continuous.

31. $f(x) = \dfrac{1}{x - 3}$

32. $g(x) = \dfrac{1}{\sqrt{x - 1}}$

In Exercises 33–36, use Theorem 7 to show that the given function is continuous.

33. $f(x) = \sqrt{\left(\dfrac{x}{x + 1}\right)}$

34. $f(x) = \sin (x^2 + 1)$

35. $f(x) = \cos (\sqrt[3]{1 - x})$

36. $f(x) = \tan \left(\dfrac{x^2}{x^2 + 4}\right)$

Group Activity In Exercises 37–40, verify that the function is continuous and state its domain. Indicate which theorems you are using, and which functions you are assuming to be continuous.

37. $y = \dfrac{1}{\sqrt{x + 2}}$

38. $y = x^2 + \sqrt[3]{4 - x}$

39. $y = |x^2 - 4x|$

40. $y = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$

In Exercises 41–44, sketch a possible graph for a function f that has the stated properties.

41. $f(3)$ exists but $\lim_{x \to 3} f(x)$ does not.

42. $f(-2)$ exists, $\lim_{x \to -2^+} f(x) = f(-2)$, but $\lim_{x \to -2} f(x)$ does not exist.

43. $f(4)$ exists, $\lim_{x \to 4} f(x)$ exists, but f is not continuous at $x = 4$.

44. $f(x)$ is continuous for all x except $x = 1$, where f has a nonremovable discontinuity.

45. ***Solving Equations*** Is any real number exactly 1 less than its fourth power? Give any such values accurate to 3 decimal places.

46. ***Solving Equations*** Is any real number exactly 2 more than its cube? Give any such values accurate to 3 decimal places.

47. ***Continuous Function*** Find a value for a so that the function

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$$

is continuous.

48. ***Continuous Function*** Find a value for a so that the function

$$f(x) = \begin{cases} 2x + 3, & x \leq 2 \\ ax + 1, & x > 2 \end{cases}$$

is continuous.

49. ***Continuous Function*** Find a value for a so that the function

$$f(x) = \begin{cases} 4 - x^2, & x < -1 \\ ax^2 - 1, & x \geq -1 \end{cases}$$

is continuous.

50. ***Continuous Function*** Find a value for a so that the function

$$f(x) = \begin{cases} x^2 + x + a, & x < 1 \\ x^3, & x \geq 1 \end{cases}$$

is continuous.

51. **Writing to Learn** Explain why the equation $e^{-x} = x$ has at least one solution.

52. ***Salary Negotiation*** A welder's contract promises a 3.5% salary increase each year for 4 years and Luisa has an initial salary of $36,500.

(a) Show that Luisa's salary is given by

$$y = 36{,}500(1.035)^{\text{int } t},$$

where t is the time, measured in years, since Luisa signed the contract.

(b) Graph Luisa's salary function. At what values of t is it continuous?

53. ***Airport Parking*** Valuepark charge $1.10 per hour or fraction of an hour for airport parking. The maximum charge per day is $7.25.

(a) Write a formula that gives the charge for x hours with $0 \leq x \leq 24$. (*Hint:* See Exercise 52.)

(b) Graph the function in part (a). At what values of x is it continuous?

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

54. **True or False** A continuous function cannot have a point of discontinuity. Justify your answer.

55. **True or False** It is possible to extend the definition of a function f at a jump discontinuity $x = a$ so that f is continuous at $x = a$. Justify your answer.

56. Multiple Choice On which of the following intervals is $f(x) = \dfrac{1}{\sqrt{x}}$ not continuous?

(A) $(0, \infty)$ **(B)** $[0, \infty)$ **(C)** $(0, 2)$

(D) $(1, 2)$ **(E)** $[1, \infty)$

57. Multiple Choice Which of the following points is not a point of discontinuity of $f(x) = \sqrt{x - 1}$?

(A) $x = -1$ **(B)** $x = -1/2$ **(C)** $x = 0$

(D) $x = 1/2$ **(E)** $x = 1$

58. Multiple Choice Which of the following statements about the function

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -x + 3, & 1 < x < 2 \end{cases}$$

is not true?

(A) $f(1)$ does not exist.

(B) $\lim_{x\to 0^+} f(x)$ exists.

(C) $\lim_{x\to 2^-} f(x)$ exists.

(D) $\lim_{x\to 1} f(x)$ exists.

(E) $\lim_{x\to 1} f(x) \neq f(1)$

59. Multiple Choice Which of the following points of discontinuity of

$$f(x) = \frac{x(x-1)(x-2)^2(x+1)^2(x-3)^2}{x(x-1)(x-2)(x+1)^2(x-3)^3}$$

is not removable?

(A) $x = -1$ **(B)** $x = 0$ **(C)** $x = 1$

(D) $x = 2$ **(E)** $x = 3$

Exploration

60. Let $f(x) = \left(1 + \dfrac{1}{x}\right)^x$.

(a) Find the domain of f. **(b)** Draw the graph of f.

(c) Writing to Learn Explain why $x = -1$ and $x = 0$ are points of discontinuity of f.

(d) Writing to Learn Are either of the discontinuities in part (c) removable? Explain.

(e) Use graphs and tables to estimate $\lim_{x\to\infty} f(x)$.

Extending the Ideas

61. *Continuity at a Point* Show that $f(x)$ is continuous at $x = a$ if and only if

$$\lim_{h\to 0} f(a + h) = f(a).$$

62. *Continuity on Closed Intervals* Let f be continuous and never zero on $[a, b]$. Show that either $f(x) > 0$ for all x in $[a, b]$ or $f(x) < 0$ for all x in $[a, b]$.

63. *Properties of Continuity* Prove that if f is continuous on an interval, then so is $|f|$.

64. *Everywhere Discontinuous* Give a convincing argument that the following function is not continuous at any real number.

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

2.4 Rates of Change and Tangent Lines

What you'll learn about

- Average Rates of Change
- Tangent to a Curve
- Slope of a Curve
- Normal to a Curve
- Speed Revisited

. . . and why

The tangent line determines the direction of a body's motion at every point along its path.

Average Rates of Change

We encounter average rates of change in such forms as average speed (in miles per hour), growth rates of populations (in percent per year), and average monthly rainfall (in inches per month). The **average rate of change** of a quantity over a period of time is the amount of change divided by the time it takes. In general, the *average rate of change* of a function over an interval is the amount of change divided by the length of the interval.

EXAMPLE 1 Finding Average Rate of Change

Find the average rate of change of $f(x) = x^3 - x$ over the interval $[1, 3]$.

SOLUTION

Since $f(1) = 0$ and $f(3) = 24$, the average rate of change over the interval $[1, 3]$ is

$$\frac{f(3) - f(1)}{3 - 1} = \frac{24 - 0}{2} = 12.$$

Now try Exercise 1.

Experimental biologists often want to know the rates at which populations grow under controlled laboratory conditions. Figure 2.27 shows how the number of fruit flies *(Drosophila)* grew in a controlled 50-day experiment. The graph was made by counting flies at regular intervals, plotting a point for each count, and drawing a smooth curve through the plotted points.

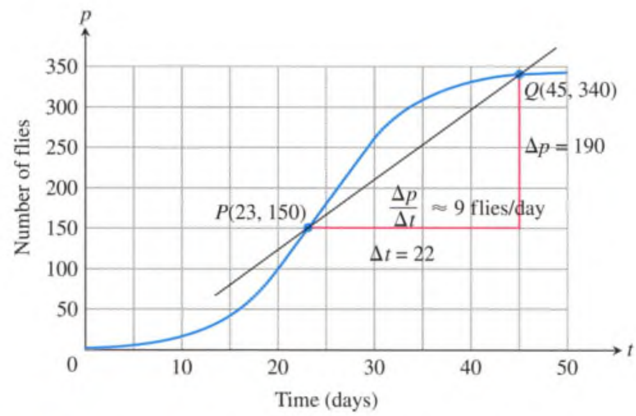


Figure 2.27 Growth of a fruit fly population in a controlled experiment.
Source: Elements of Mathematical Biology. (Example 2)

Secant to a Curve

A line through two points on a curve is a **secant to the curve.**

EXAMPLE 2 Growing Drosophila in a Laboratory

Use the points $P(23, 150)$ and $Q(45, 340)$ in Figure 2.27 to compute the average rate of change and the slope of the secant line PQ.

SOLUTION

There were 150 flies on day 23 and 340 flies on day 45. This gives an increase of $340 - 150 = 190$ flies in $45 - 23 = 22$ days.

The average rate of change in the population p from day 23 to day 45 was

$$\textit{Average rate of change: } \frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies/day},$$

or about 9 flies per day.

continued

Marjorie Lee Browne
(1914–1979)

When Marjorie Browne graduated from the University of Michigan in 1949, she was one of the first two African American women to be awarded a Ph.D. in Mathematics. Browne went on to become chairperson of the mathematics department at North Carolina Central University, and succeeded in obtaining grants for retraining high school mathematics teachers.

This average rate of change is also the slope of the secant line through the two points P and Q on the population curve. We can calculate the slope of the secant PQ from the coordinates of P and Q.

$$\textit{Secant slope: } \frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies/day}$$

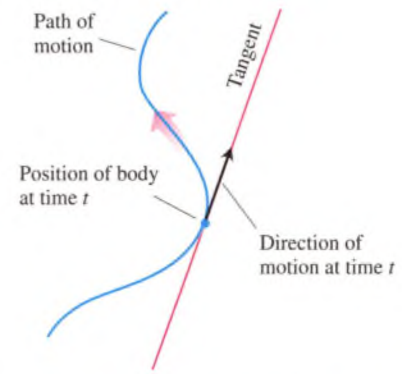
Now try Exercise 7.

As suggested by Example 2, *we can always think of an average rate of change as the slope of a secant line.*

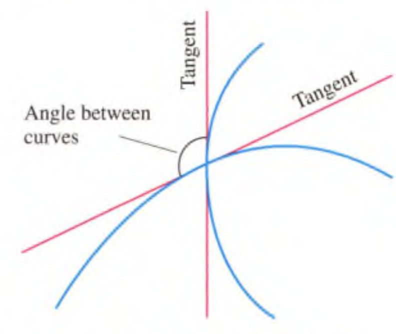
In addition to knowing the average rate at which the population grew from day 23 to day 45, we may also want to know how fast the population was growing on day 23 itself. To find out, we can watch the slope of the secant PQ change as we back Q along the curve toward P. The results for four positions of Q are shown in Figure 2.28.

Why Find Tangents to Curves?

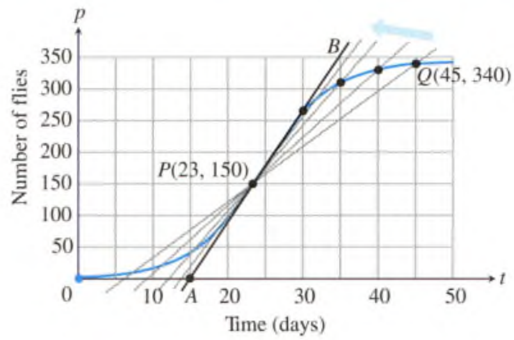
In mechanics, the tangent determines the direction of a body's motion at every point along its path.



In geometry, the tangents to two curves at a point of intersection determine the angle at which the curves intersect.



In optics, the tangent determines the angle at which a ray of light enters a curved lens (more about this in Section 3.7). The problem of how to find a tangent to a curve became the dominant mathematical problem of the early seventeenth century and it is hard to overestimate how badly the scientists of the day wanted to know the answer. Descartes went so far as to say that the problem was the most useful and most general problem not only that he knew but that he had any desire to know.



(a)

Q	Slope of $PQ = \Delta p/\Delta t$ (flies/day)
(45, 340)	$(340 - 150)/(45 - 23) \approx 8.6$
(40, 330)	$(330 - 150)/(40 - 23) \approx 10.6$
(35, 310)	$(310 - 150)/(35 - 23) \approx 13.3$
(30, 265)	$(265 - 150)/(30 - 23) \approx 16.4$

(b)

Figure 2.28 (a) Four secants to the fruit fly graph of Figure 2.27, through the point $P(23, 150)$. (b) The slopes of the four secants.

In terms of geometry, what we see as Q approaches P along the curve is this: The secant PQ approaches the tangent line AB that we drew by eye at P. This means that within the limitations of our drawing, the slopes of the secants approach the slope of the tangent, which we calculate from the coordinates of A and B to be

$$\frac{350 - 0}{35 - 15} = 17.5 \text{ flies/day.}$$

In terms of population, what we see as Q approaches P is this: The average growth rates for increasingly smaller time intervals approach the slope of the tangent to the curve at P (17.5 flies per day). The slope of the tangent line is therefore the number we take as the rate at which the fly population was growing on day $t = 23$.

Tangent to a Curve

The moral of the fruit fly story would seem to be that we should define the rate at which the value of the function $y = f(x)$ is changing with respect to x at any particular value $x = a$ to be the slope of the tangent to the curve $y = f(x)$ at $x = a$. But how are we to define the tangent line at an arbitrary point P on the curve and find its slope from the formula $y = f(x)$? The problem here is that we know only one point. Our usual definition of slope requires two points.

The solution that mathematician Pierre Fermat found in 1629 proved to be one of that century's major contributions to calculus. We still use his method of defining tangents to produce formulas for slopes of curves and rates of change:

1. We start with what we can calculate, namely, the slope of a secant through P and a point Q nearby on the curve.

2. We find the limiting value of the secant slope (if it exists) as Q approaches P along the curve.

3. We define the *slope of the curve* at P to be this number and define the *tangent to the curve* at P to be the line through P with this slope.

EXAMPLE 3 Finding Slope and Tangent Line

Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$. Write an equation for the tangent to the parabola at this point.

SOLUTION

We begin with a secant line through $P(2, 4)$ and a nearby point $Q(2 + h, (2 + h)^2)$ on the curve (Figure 2.29).

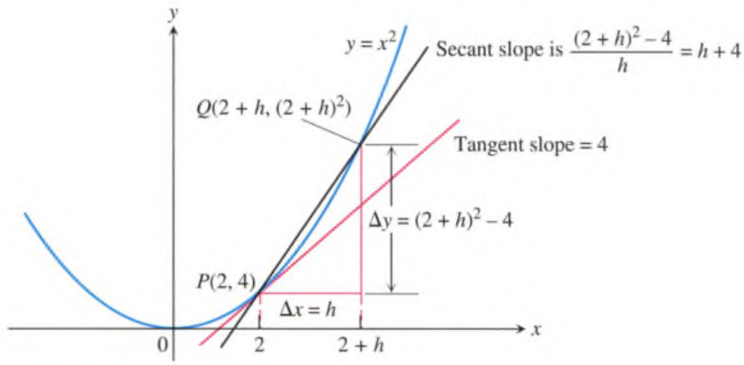


Figure 2.29 The slope of the tangent to the parabola $y = x^2$ at $P(2, 4)$ is 4.

We then write an expression for the slope of the secant line and find the limiting value of this slope as Q approaches P along the curve.

$$\text{Secant slope} = \frac{\Delta y}{\Delta x} = \frac{(2 + h)^2 - 4}{h}$$
$$= \frac{h^2 + 4h + 4 - 4}{h}$$
$$= \frac{h^2 + 4h}{h} = h + 4$$

The limit of the secant slope as Q approaches P along the curve is

$$\lim_{Q \to P} (\text{secant slope}) = \lim_{h \to 0} (h + 4) = 4.$$

Thus, the slope of the parabola at P is 4.

The tangent to the parabola at P is the line through $P(2, 4)$ with slope $m = 4$.

$$y - 4 = 4(x - 2)$$
$$y = 4x - 8 + 4$$
$$y = 4x - 4$$

Now try Exercise 11 (a, b).

Slope of a Curve

To find the tangent to a curve $y = f(x)$ at a point $P(a, f(a))$ we use the same dynamic procedure. We calculate the slope of the secant line through P and a point $Q(a + h, f(a + h))$. We then investigate the limit of the slope as $h \to 0$ (Figure 2.30). If the limit exists, it is the slope of the curve at P and we define the tangent at P to be the line through P having this slope.

Pierre de Fermat
(1601–1665)

The dynamic approach to tangency, invented by Fermat in 1629, proved to be one of the seventeenth century's major contributions to calculus.

Fermat, a skilled linguist and one of his century's greatest mathematicians, tended to confine his writing to professional correspondence and to papers written for personal friends. He rarely wrote completed descriptions of his work, even for his personal use. His name slipped into relative obscurity until the late 1800s, and it was only from a four-volume edition of his works published at the beginning of this century that the true importance of his many achievements became clear.

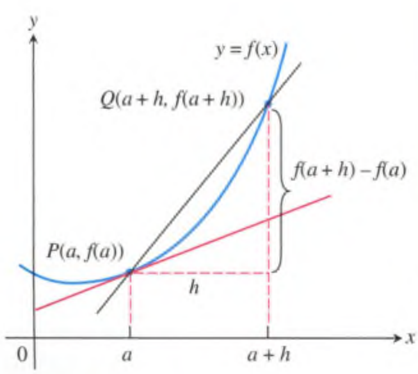


Figure 2.30 The tangent slope is

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

DEFINITION Slope of a Curve at a Point

The **slope of the curve** $y = f(x)$ at the point $P(a, f(a))$ is the number

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists.

The **tangent line to the curve** at P is the line through P with this slope.

EXAMPLE 4 Exploring Slope and Tangent

Let $f(x) = 1/x$.

(a) Find the slope of the curve at $x = a$.
(b) Where does the slope equal $-1/4$?
(c) What happens to the tangent to the curve at the point $(a, 1/a)$ for different values of a?

SOLUTION

(a) The slope at $x = a$ is

$$\begin{aligned}\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h \to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} \\ &= \lim_{h \to 0} \frac{1}{h} \cdot \frac{a - (a+h)}{a(a+h)} \\ &= \lim_{h \to 0} \cdot \frac{-h}{ha(a+h)} \\ &= \lim_{h \to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}.\end{aligned}$$

(b) The slope will be $-1/4$ if

$$-\frac{1}{a^2} = -\frac{1}{4}$$

$$a^2 = 4 \qquad \text{Multiply by } -4a^2.$$

$$a = \pm 2.$$

The curve has the slope $-1/4$ at the two points $(2, 1/2)$ and $(-2, -1/2)$ (Figure 2.31).

(c) The slope $-1/a^2$ is always negative. As $a \to 0^+$, the slope approaches $-\infty$ and the tangent becomes increasingly steep. We see this again as $a \to 0^-$. As a moves away from the origin in either direction, the slope approaches 0 and the tangent becomes increasingly horizontal. *Now try Exercise 19.*

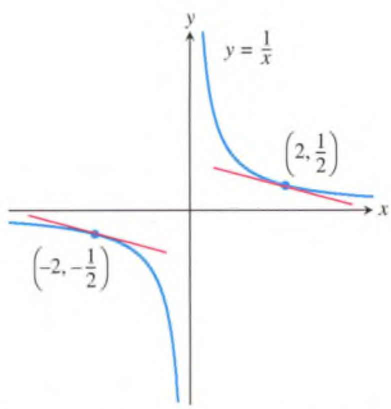


Figure 2.31 The two tangent lines to $y = 1/x$ having slope $-1/4$. (Example 4)

The expression

$$\frac{f(a+h) - f(a)}{h}$$

is the **difference quotient of f at a.** Suppose the difference quotient has a limit as h approaches zero. If we interpret the difference quotient as a secant slope, the limit is the slope of both the curve and the tangent to the curve at the point $x = a$. If we interpret the difference quotient as an average rate of change, the limit is the function's rate of change with respect to x at the point $x = a$. This limit is one of the two most important mathematical objects considered in calculus. We will begin a thorough study of it in Chapter 3.

All of these are the same:

1. the slope of $y = f(x)$ at $x = a$
2. the slope of the tangent to $y = f(x)$ at $x = a$
3. the (instantaneous) rate of change of $f(x)$ with respect to x at $x = a$
4. $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

About the Word *Normal*

When analytic geometry was developed in the seventeenth century, European scientists still wrote about their work and ideas in Latin, the one language that all educated Europeans could read and understand. The Latin word *normalis,* which scholars used for *perpendicular,* became *normal* when they discussed geometry in English.

Normal to a Curve

The **normal line** to a curve at a point is the line perpendicular to the tangent at that point.

EXAMPLE 5 Finding a Normal Line

Write an equation for the normal to the curve $f(x) = 4 - x^2$ at $x = 1$.

SOLUTION

The slope of the tangent to the curve at $x = 1$ is

$$\begin{aligned}\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0} \frac{4-(1+h)^2-3}{h}\\ &= \lim_{h\to 0} \frac{4-1-2h-h^2-3}{h}\\ &= \lim_{h\to 0} \frac{-h(2+h)}{h} = -2.\end{aligned}$$

Thus, the slope of the normal is $1/2$, the negative reciprocal of -2. The normal to the curve at $(1, f(1)) = (1, 3)$ is the line through $(1, 3)$ with slope $m = 1/2$.

$$\begin{aligned} y - 3 &= \frac{1}{2}(x-1)\\ y &= \frac{1}{2}x - \frac{1}{2} + 3\\ y &= \frac{1}{2}x + \frac{5}{2}\end{aligned}$$

You can support this result by drawing the graphs in a square viewing window.

Now try Exercise 11 (c, d).

Speed Revisited

The function $y = 16t^2$ that gave the distance fallen by the rock in Example 1, Section 2.1, was the rock's *position function.* A body's average speed along a coordinate axis (here, the y-axis) for a given period of time is the average rate of change of its *position* $y = f(t)$. Its **instantaneous speed** at any time t is the **instantaneous rate of change** of position with respect to time at time t, or

$$\lim_{h\to 0} \frac{f(t+h)-f(t)}{h}.$$

We saw in Example 1, Section 2.1, that the rock's instantaneous speed at $t = 2$ sec was 64 ft/sec.

Particle Motion

We only have considered objects moving in one direction in this chapter. In Chapter 3, we will deal with more complicated motion.

EXAMPLE 6 Investigating Free Fall

Find the speed of the falling rock in Example 1, Section 2.1, at $t = 1$ sec.

SOLUTION

The position function of the rock is $f(t) = 16t^2$. The average speed of the rock over the interval between $t = 1$ and $t = 1 + h$ sec was

$$\frac{f(1+h)-f(1)}{h} = \frac{16(1+h)^2 - 16(1)^2}{h} = \frac{16(h^2+2h)}{h} = 16(h+2).$$

The rock's speed at the instant $t = 1$ was

$$\lim_{h\to 0} 16(h+2) = 32 \text{ ft/sec}.$$

Now try Exercise 27.

Quick Review 2.4 *(For help, go to Section 1.1.)*

In Exercises 1 and 2, find the increments Δx and Δy from point A to point B.

1. $A(-5, 2)$, $B(3, 5)$ **2.** $A(1, 3)$, $B(a, b)$

In Exercises 3 and 4, find the slope of the line determined by the points.

3. $(-2, 3)$, $(5, -1)$ **4.** $(-3, -1)$, $(3, 3)$

In Exercises 5–9, write an equation for the specified line.

5. through $(-2, 3)$ with slope $= 3/2$

6. through $(1, 6)$ and $(4, -1)$

7. through $(1, 4)$ and parallel to $y = -\frac{3}{4}x + 2$

8. through $(1, 4)$ and perpendicular to $y = -\frac{3}{4}x + 2$

9. through $(-1, 3)$ and parallel to $2x + 3y = 5$

10. For what value of b will the slope of the line through $(2, 3)$ and $(4, b)$ be $5/3$?

Section 2.4 Exercises

In Exercises 1–6, find the average rate of change of the function over each interval.

1. $f(x) = x^3 + 1$
 (a) $[2, 3]$ **(b)** $[-1, 1]$

2. $f(x) = \sqrt{4x + 1}$
 (a) $[0, 2]$ **(b)** $[10, 12]$

3. $f(x) = e^x$
 (a) $[-2, 0]$ **(b)** $[1, 3]$

4. $f(x) = \ln x$
 (a) $[1, 4]$ **(b)** $[100, 103]$

5. $f(x) = \cot t$
 (a) $[\pi/4, 3\pi/4]$ **(b)** $[\pi/6, \pi/2]$

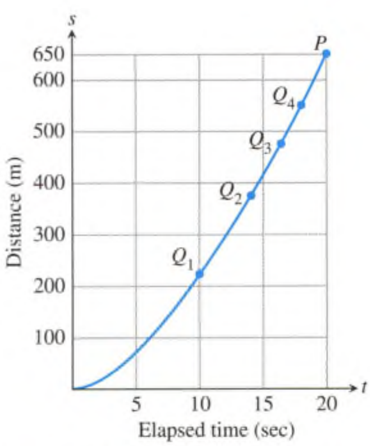
6. $f(x) = 2 + \cos t$
 (a) $[0, \pi]$ **(b)** $[-\pi, \pi]$

In Exercises 7 and 8, a distance-time graph is shown.

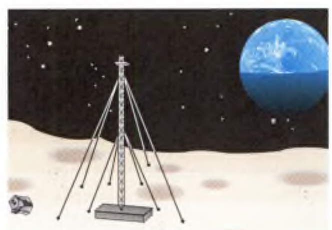
(a) Estimate the slopes of the secants PQ_1, PQ_2, PQ_3, and PQ_4, arranging them in order in a table. What is the appropriate unit for these slopes?

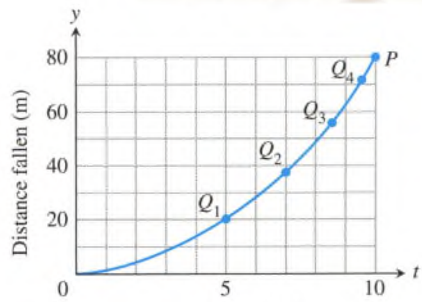
(b) Estimate the speed at point P.

7. ***Accelerating from a Standstill*** The figure shows the distance-time graph for a 1994 Ford® Mustang Cobra™ accelerating from a standstill.



8. ***Lunar Data*** The accompanying figure shows a distance-time graph for a wrench that fell from the top platform of a communication mast on the moon to the station roof 80 m below.





In Exercises 9–12, at the indicated point find

(a) the slope of the curve,

(b) an equation of the tangent, and

(c) an equation of the normal.

(d) Then draw a graph of the curve, tangent line, and normal line in the same square viewing window.

9. $y = x^2$ at $x = -2$ **10.** $y = x^2 - 4x$ at $x = 1$

11. $y = \dfrac{1}{x - 1}$ at $x = 2$ **12.** $y = x^2 - 3x - 1$ at $x = 0$

In Exercises 13 and 14, find the slope of the curve at the indicated point.

13. $f(x) = |x|$ at **(a)** $x = 2$ **(b)** $x = -3$

14. $f(x) = |x - 2|$ at $x = 1$

In Exercises 15–18, determine whether the curve has a tangent at the indicated point. If it does, give its slope. If not, explain why not.

15. $f(x) = \begin{cases} 2 - 2x - x^2, & x < 0 \\ 2x + 2, & x \ge 0 \end{cases}$ at $x = 0$

16. $f(x) = \begin{cases} -x, & x < 0 \\ x^2 - x, & x \ge 0 \end{cases}$ at $x = 0$

17. $f(x) = \begin{cases} 1/x, & x \le 2 \\ \dfrac{4-x}{4}, & x > 2 \end{cases}$ at $x = 2$

18. $f(x) = \begin{cases} \sin x, & 0 \le x < 3\pi/4 \\ \cos x, & 3\pi/4 \le x \le 2\pi \end{cases}$ at $x = 3\pi/4$

In Exercises 19–22, **(a)** find the slope of the curve at $x = a$.
(b) Writing to Learn Describe what happens to the tangent at $x = a$ as a changes.

19. $y = x^2 + 2$

20. $y = 2/x$

21. $y = \dfrac{1}{x-1}$

22. $y = 9 - x^2$

23. ***Free Fall*** An object is dropped from the top of a 100-m tower. Its height above ground after t sec is $100 - 4.9t^2$ m. How fast is it falling 2 sec after it is dropped?

24. ***Rocket Launch*** At t sec after lift-off, the height of a rocket is $3t^2$ ft. How fast is the rocket climbing after 10 sec?

25. ***Area of Circle*** What is the rate of change of the area of a circle with respect to the radius when the radius is $r = 3$ in.?

26. ***Volume of Sphere*** What is the rate of change of the volume of a sphere with respect to the radius when the radius is $r = 2$ in.?

27. ***Free Fall on Mars*** The equation for free fall at the surface of Mars is $s = 1.86t^2$ m with t in seconds. Assume a rock is dropped from the top of a 200-m cliff. Find the speed of the rock at $t = 1$ sec.

28. ***Free Fall on Jupiter*** The equation for free fall at the surface of Jupiter is $s = 11.44t^2$ m with t in seconds. Assume a rock is dropped from the top of a 500-m cliff. Find the speed of the rock at $t = 2$ sec.

29. ***Horizontal Tangent*** At what point is the tangent to $f(x) = x^2 + 4x - 1$ horizontal?

30. ***Horizontal Tangent*** At what point is the tangent to $f(x) = 3 - 4x - x^2$ horizontal?

31. ***Finding Tangents and Normals***

(a) Find an equation for each tangent to the curve $y = 1/(x-1)$ that has slope -1. (See Exercise 21.)

(b) Find an equation for each normal to the curve $y = 1/(x-1)$ that has slope 1.

32. ***Finding Tangents*** Find the equations of all lines tangent to $y = 9 - x^2$ that pass through the point (1, 12).

33. Table 2.2 gives the amount of federal spending in billions of dollars for national defense for several years.

Table 2.2 National Defense Spending

Year	National Defense Spending ($ billions)
1990	299.3
1995	272.1
1999	274.9
2000	294.5
2001	305.5
2002	348.6
2003	404.9

Source: U.S. Census Bureau, *Statistical Abstract of the United States, 2004–2005.*

(a) Find the average rate of change in spending from 1990 to 1995.

(b) Find the average rate of change in spending from 2000 to 2001.

(c) Find the average rate of change in spending from 2002 to 2003.

(d) Let $x = 0$ represent 1990, $x = 1$ represent 1991, and so forth. Find the quadratic regression equation for the data and superimpose its graph on a scatter plot of the data.

(e) Compute the average rates of change in parts (a), (b), and (c) using the regression equation.

(f) Use the regression equation to find how fast the spending was growing in 2003.

(g) Writing to Learn Explain why someone might be hesitant to make predictions about the rate of change of national defense spending based on this equation.

34. Table 2.3 gives the amount of federal spending in billions of dollars for agriculture for several years.

Table 2.3 Agriculture Spending

Year	Agriculture Spending ($ billions)
1990	12.0
1995	9.8
1999	23.0
2000	36.6
2001	26.4
2002	22.0
2003	22.6

Source: U.S. Census Bureau, *Statistical Abstract of the United States, 2004–2005.*

(a) Let $x = 0$ represent 1990, $x = 1$ represent 1991, and so forth. Make a scatter plot of the data.

(b) Let P represent the point corresponding to 2003, Q_1 the point corresponding to 2000, Q_2 the point corresponding to 2001, and Q_3 the point corresponding to 2002. Find the slope of the secant line PQ_i for $i = 1, 2, 3$.

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

35. True or False If the graph of a function has a tangent line at $x = a$, then the graph also has a normal line at $x = a$. Justify your answer.

36. True or False The graph of $f(x)=|x|$ has a tangent line at $x = 0$. Justify your answer.

37. Multiple Choice If the line L tangent to the graph of a function f at the point (2, 5) passes through the point $(-1, -3)$, what is the slope of L?

(A) $-3/8$ **(B)** $3/8$ **(C)** $-8/3$ **(D)** $8/3$ **(E)** undefined

38. Multiple Choice Find the average rate of change of $f(x) = x^2 + x$ over the interval [1, 3].

(A) -5 **(B)** $1/5$ **(C)** $1/4$ **(D)** 4 **(E)** 5

39. Multiple Choice Which of the following is an equation of the tangent to the graph of $f(x) = 2/x$ at $x = 1$?

(A) $y = -2x$ **(B)** $y = 2x$ **(C)** $y = -2x + 4$
(D) $y = -x + 3$ **(E)** $y = x + 3$

40. Multiple Choice Which of the following is an equation of the normal to the graph of $f(x) = 2/x$ at $x = 1$?

(A) $y = \frac{1}{2}x + \frac{3}{2}$ **(B)** $y = -\frac{1}{2}x$ **(C)** $y = \frac{1}{2}x + 2$
(D) $y = -\frac{1}{2}x + 2$ **(E)** $y = 2x + 5$

Explorations

In Exercises 41 and 42, complete the following for the function.

(a) Compute the difference quotient

$$\frac{f(1 + h) - f(1)}{h}.$$

(b) Use graphs and tables to estimate the limit of the difference quotient in part (a) as $h \to 0$.

(c) Compare your estimate in part (b) with the given number.

(d) Writing to Learn Based on your computations, do you think the graph of f has a tangent at $x = 1$? If so, estimate its slope. If not, explain why not.

41. $f(x) = e^x$, e **42.** $f(x) = 2^x$, $\ln 4$

Group Activity In Exercises 43–46, the curve $y = f(x)$ has a **vertical tangent** at $x = a$ if

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \infty$$

or if

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = -\infty.$$

In each case, the right- and left-hand limits are required to be the same: both $+\infty$ or both $-\infty$.

Use graphs to investigate whether the curve has a vertical tangent at $x = 0$.

43. $y = x^{2/5}$ **44.** $y = x^{3/5}$
45. $y = x^{1/3}$ **46.** $y = x^{2/3}$

Extending the Ideas

In Exercises 47 and 48, determine whether the graph of the function has a tangent at the origin. Explain your answer.

47. $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

48. $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

49. *Sine Function* Estimate the slope of the curve $y = \sin x$ at $x = 1$. (*Hint:* See Exercises 41 and 42.)

Quick Quiz for AP* Preparation: Sections 2.3 and 2.4

You may use a calculator with these problems.

1. Multiple Choice Which of the following values is the average rate of $f(x) = \sqrt{x + 1}$ over the interval (0, 3)?

(A) -3 **(B)** -1 **(C)** $-1/3$ **(D)** $1/3$ **(E)** 3

2. Multiple Choice Which of the following statements is false for the function

$$f(x) = \begin{cases} \frac{3}{4}x, & 0 \le x < 4 \\ 2, & x = 4 \\ -x + 7, & 4 < x \le 6 \\ 1, & 6 < x < 8? \end{cases}$$

(A) $\lim_{x \to 4} f(x)$ exists **(B)** $f(4)$ exists
(C) $\lim_{x \to 6} f(x)$ exists **(D)** $\lim_{x \to 8^-} f(x)$ exists
(E) f is continuous at $x = 4$

3. Multiple Choice Which of the following is an equation for the tangent line to $f(x) = 9 - x^2$ at $x = 2$?

(A) $y = \frac{1}{4}x + \frac{9}{2}$ **(B)** $y = -4x + 13$
(C) $y = -4x - 3$ **(D)** $y = 4x - 3$
(E) $y = 4x + 13$

4. Free Response Let $f(x) = 2x - x^2$.

(a) Find $f(3)$. **(b)** Find $f(3 + h)$.

(c) Find $\frac{f(3 + h) - f(3)}{h}$.

(d) Find the instantaneous rate of change of f at $x = 3$.

Chapter 2 Key Terms

average rate of change (p. 87)
average speed (p. 59)
connected graph (p. 83)
Constant Multiple Rule for Limits (p. 61)
continuity at a point (p. 78)
continuous at an endpoint (p. 79)
continuous at an interior point (p. 79)
continuous extension (p. 81)
continuous function (p. 81)
continuous on an interval (p. 81)
difference quotient (p. 90)
Difference Rule for Limits (p. 61)
discontinuous (p. 79)
end behavior model (p. 74)
free fall (p. 91)
horizontal asymptote (p. 70)
infinite discontinuity (p. 80)
instantaneous rate of change (p. 91)
instantaneous speed (p. 91)
intermediate value property (p. 83)
Intermediate Value Theorem for Continuous Functions (p. 83)
jump discontinuity (p. 80)
left end behavior model (p. 74)
left-hand limit (p. 64)
limit of a function (p. 60)
normal to a curve (p. 91)
oscillating discontinuity (p. 80)
point of discontinuity (p. 79)
Power Rule for Limits (p. 71)
Product Rule for Limits (p. 61)
Properties of Continuous Functions (p. 82)
Quotient Rule for Limits (p. 61)
removable discontinuity (p. 80)
right end behavior model (p. 74)
right-hand limit (p. 64)
Sandwich Theorem (p. 65)
secant to a curve (p. 87)
slope of a curve (p. 89)
Sum Rule for Limits (p. 61)
tangent line to a curve (p. 88)
two-sided limit (p. 64)
vertical asymptote (p. 72)
vertical tangent (p. 94)

Chapter 2 Review Exercises

The collection of exercises marked in red could be used as a chapter test.

In Exercises 1–14, find the limits.

1. $\lim_{x\to-2} (x^3 - 2x^2 + 1)$

2. $\lim_{x\to-2} \dfrac{x^2+1}{3x^2-2x+5}$

3. $\lim_{x\to4} \sqrt{1-2x}$

4. $\lim_{x\to5} \sqrt[4]{9-x^2}$

5. $\lim_{x\to0} \dfrac{\frac{1}{2+x}-\frac{1}{2}}{x}$

6. $\lim_{x\to\pm\infty} \dfrac{2x^2+3}{5x^2+7}$

7. $\lim_{x\to\pm\infty} \dfrac{x^4+x^3}{12x^3+128}$

8. $\lim_{x\to0} \dfrac{\sin 2x}{4x}$

9. $\lim_{x\to0} \dfrac{x\csc x+1}{x\csc x}$

10. $\lim_{x\to0} e^x \sin x$

11. $\lim_{x\to7/2^+} \text{int}\,(2x-1)$

12. $\lim_{x\to7/2^-} \text{int}\,(2x-1)$

13. $\lim_{x\to\infty} e^{-x}\cos x$

14. $\lim_{x\to\infty} \dfrac{x+\sin x}{x+\cos x}$

In Exercises 15–20, determine whether the limit exists on the basis of the graph of $y=f(x)$. The domain of f is the set of real numbers.

15. $\lim_{x\to d} f(x)$

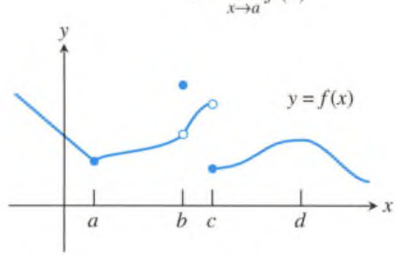
16. $\lim_{x\to c^+} f(x)$

17. $\lim_{x\to c^-} f(x)$

18. $\lim_{x\to c} f(x)$

19. $\lim_{x\to b} f(x)$

20. $\lim_{x\to a} f(x)$



In Exercises 21–24, determine whether the function f used in Exercises 15–20 is continuous at the indicated point.

21. $x=a$

22. $x=b$

23. $x=c$

24. $x=d$

In Exercises 25 and 26, use the graph of the function with domain $-1 \le x \le 3$.

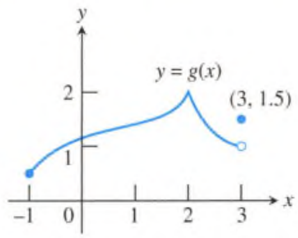
25. Determine

(a) $\lim_{x \to 3^-} g(x)$. **(b)** $g(3)$.

(c) whether $g(x)$ is continuous at $x = 3$.

(d) the points of discontinuity of $g(x)$.

(e) Writing to Learn whether any points of discontinuity are removable. If so, describe the new function. If not, explain why not.



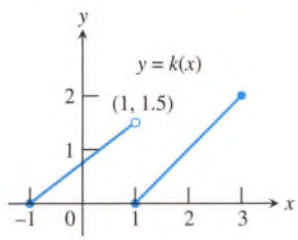
26. Determine

(a) $\lim_{x \to 1^-} k(x)$. **(b)** $\lim_{x \to 1^+} k(x)$. **(c)** $k(1)$.

(d) whether $k(x)$ is continuous at $x = 1$.

(e) the points of discontinuity of $k(x)$.

(f) Writing to Learn whether any points of discontinuity are removable. If so, describe the new function. If not, explain why not.



In Exercises 27 and 28, **(a)** find the vertical asymptotes of the graph of $y = f(x)$, and **(b)** describe the behavior of $f(x)$ to the left and right of any vertical asymptote.

27. $f(x) = \dfrac{x+3}{x+2}$

28. $f(x) = \dfrac{x-1}{x^2\,(x+2)}$

In Exercises 29 and 30, answer the questions for the piecewise-defined function.

29. $f(x) = \begin{cases} 1, & x \le -1 \\ -x, & -1 < x < 0 \\ 1, & x = 0 \\ -x, & 0 < x < 1 \\ 1, & x \ge 1 \end{cases}$

(a) Find the right-hand and left-hand limits of f at $x = -1$, 0, and 1.

(b) Does f have a limit as x approaches -1? 0? 1? If so, what is it? If not, why not?

(c) Is f continuous at $x = -1$? 0? 1? Explain.

30. $f(x) = \begin{cases} |x^3 - 4x|, & x < 1 \\ x^2 - 2x - 2, & x \ge 1 \end{cases}$

(a) Find the right-hand and left-hand limits of f at $x = 1$.

(b) Does f have a limit as $x \to 1$? If so, what is it? If not, why not?

(c) At what points is f continuous?

(d) At what points is f discontinuous?

In Exercises 31 and 32, find all points of discontinuity of the function.

31. $f(x) = \dfrac{x+1}{4-x^2}$

32. $g(x) = \sqrt[3]{3x+2}$

In Exercises 33–36, find **(a)** a power function end behavior model and **(b)** any horizontal asymptotes.

33. $f(x) = \dfrac{2x+1}{x^2 - 2x + 1}$

34. $f(x) = \dfrac{2x^2 + 5x - 1}{x^2 + 2x}$

35. $f(x) = \dfrac{x^3 - 4x^2 + 3x + 3}{x - 3}$

36. $f(x) = \dfrac{x^4 - 3x^2 + x - 1}{x^3 - x + 1}$

In Exercises 37 and 38, find **(a)** a right end behavior model and **(b)** a left end behavior model for the function.

37. $f(x) = x + e^x$

38. $f(x) = \ln |x| + \sin x$

Group Activity In Exercises 39 and 40, what value should be assigned to k to make f a continuous function?

39. $f(x) = \begin{cases} \dfrac{x^2 + 2x - 15}{x - 3}, & x \ne 3 \\ k, & x = 3 \end{cases}$

40. $f(x) = \begin{cases} \dfrac{\sin x}{2x}, & x \ne 0 \\ k, & x = 0 \end{cases}$

Group Activity In Exercises 41 and 42, sketch a graph of a function f that satisfies the given conditions.

41. $\lim_{x \to \infty} f(x) = 3, \quad \lim_{x \to -\infty} f(x) = \infty,$

$\lim_{x \to 3^+} f(x) = \infty, \quad \lim_{x \to 3^-} f(x) = -\infty$

42. $\lim_{x \to 2} f(x)$ does not exist, $\lim_{x \to 2^+} f(x) = f(2) = 3$

43. ***Average Rate of Change*** Find the average rate of change of $f(x) = 1 + \sin x$ over the interval $[0, \pi/2]$.

44. ***Rate of Change*** Find the instantaneous rate of change of the volume $V = (1/3)\pi r^2 H$ of a cone with respect to the radius r at $r = a$ if the height H does not change.

45. ***Rate of Change*** Find the instantaneous rate of change of the surface area $S = 6x^2$ of a cube with respect to the edge length x at $x = a$.

46. ***Slope of a Curve*** Find the slope of the curve $y = x^2 - x - 2$ at $x = a$.

47. ***Tangent and Normal*** Let $f(x) = x^2 - 3x$ and $P = (1, f(1))$. Find **(a)** the slope of the curve $y = f(x)$ at P, **(b)** an equation of the tangent at P, and **(c)** an equation of the normal at P.

48. ***Horizontal Tangents*** At what points, if any, are the tangents to the graph of $f(x) = x^2 - 3x$ horizontal? (See Exercise 47.)

49. ***Bear Population*** The number of bears in a federal wildlife reserve is given by the population equation

$$p(t) = \frac{200}{1 + 7e^{-0.1t}},$$

where t is in years.

(a) Writing to Learn Find $p(0)$. Give a possible interpretation of this number.

(b) Find $\lim_{t\to\infty} p(t)$.

(c) Writing to Learn Give a possible interpretation of the result in part (b).

50. ***Taxi Fares*** Bluetop Cab charges \$3.20 for the first mile and \$1.35 for each additional mile or part of a mile.

(a) Write a formula that gives the charge for x miles with $0 \le x \le 20$.

(b) Graph the function in (a). At what values of x is it discontinuous?

51. Table 2.4 gives the population of Florida for several years.

Table 2.4 Population of Florida

Year	Population (in thousands)
1998	15,487
1999	15,759
2000	15,983
2001	16,355
2002	16,692
2003	17,019

Source: U.S. Census Bureau, *Statistical Abstract of the United States; 2004–2005.*

(a) Let $x = 0$ represent 1990, $x = 1$ represent 1991, and so forth. Make a scatter plot for the data.

(b) Let P represent the point corresponding to 2003, Q_1 the point corresponding to 1998, Q_2 the point corresponding to 1999, . . . , and Q_5 the point corresponding to 2002. Find the slope of the secant the PQ_i for $i = 1, 2, 3, 4, 5$.

(c) Predict the rate of change of population in 2003.

(d) Find a linear regression equation for the data, and use it to calculate the rate of the population in 2003.

52. ***Limit Properties*** Assume that

$$\lim_{x\to c} [f(x) + g(x)] = 2,$$

$$\lim_{x\to c} [f(x) - g(x)] = 1,$$

and that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Find $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$.

AP* Examination Preparation

You should solve the following problems without using a graphing calculation.

53. Free Response Let $f(x) = \dfrac{x}{|x^2 - 9|}$.

(a) Find the domain of f.

(b) Write an equation for each vertical asymptote of the graph of f.

(c) Write an equation for each horizontal asymptote of the graph of f.

(d) Is f odd, even, or neither? Justify your answer.

(e) Find all values of x for which f is discontinuous and classify each discontinuity as removable or nonremovable.

54. Free Response Let $f(x) = \begin{cases} x^2 - a^2x & \text{if } x < 2, \\ 4 - 2x^2 & \text{if } x \ge 2. \end{cases}$

(a) Find $\lim_{x\to 2^-} f(x)$.

(b) Find $\lim_{x\to 2^+} f(x)$.

(c) Find all values of a that make f continuous at 2. Justify your answer.

55. Free Response Let $f(x) = \dfrac{x^3 - 2x^2 + 1}{x^2 + 3}$.

(a) Find all zeros of f.

(b) Find a right end behavior model $g(x)$ for f.

(c) Determine $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} \dfrac{f(x)}{g(x)}$.